Section 7.1. Parametric Equations.

Examples. Find a few points on the curve and guess the shape of the curve. Then eliminate the parameter to obtain an equation in x and y. Confirm the shape of the curve.

1. $x(t) = t + 3$, $y(t) = 2t + 1$
2. $x(t) = t + 3$, $y(t) = t^2 + 1$, $t \geq -1$

Exercises. Find a few points on the curve and guess the shape of the curve. Then eliminate the parameter to obtain an equation in x and y. Confirm the shape of the curve.

1. $x(t) = \cos(t)$, $y(t) = \sin(t)$

   Solution. First plot a few points. Since $\cos^2(t) + \sin^2(t) = 1$, we have $x^2 + y^2 = 1$. The graph is a circle centered at the origin and with radius 1.

2. $x(t) = 4\cos(t)$, $y(t) = 3\sin(t)$

   Solution. First plot a few points. From $\cos^2(t) + \sin^2(t) = 1$, we have $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$. The graph is an ellipse centered at the origin with $a = 4$ and $b = 3$.

3. $x(t) = 2t + 1$, $y(t) = \sqrt{2t + 4}$, $-2 \leq t \leq 6$

   Solution. First plot a few points. Since $y^2 = 2t + 4 = (2t + 1) + 3 = x + 3$, we have $x = y^2 - 3$.

Section 7.2. Calculus of Parametric Curves.

Examples.

1. Find an equation of the tangent line to the curve given by $x(t) = t^2 - 3$, $y(t) = 2t - 1$ at the point corresponding to $t = 2$.

   Solution. At $t = 2$, we have $x = 1$ and $y = 3$. Also, $\dfrac{dy}{dx} = \dfrac{2}{2t} = \dfrac{1}{t} = \dfrac{1}{2}$, so the equation of the tangent line is $y - 3 = \dfrac{1}{2}(x - 1)$ or $y = \dfrac{x}{2} + \dfrac{5}{2}$.

2. Find the second derivative $\dfrac{d^2y}{dx^2}$ for $x(t) = t^2 - 3$, $y(t) = 2t - 1$ at $t = 2$.

   Solution. In the first example we calculated that $\dfrac{dy}{dx} = \dfrac{1}{t}$, so
   $\dfrac{d^2y}{dx^2} = \dfrac{d\left(\frac{dy}{dx}\right)}{dx} = \dfrac{d\left(\frac{1}{t}\right)}{dx} = \dfrac{\frac{d\left(\frac{1}{t}\right)}{dt}}{\frac{dx}{dt}} = \dfrac{-\frac{1}{t^2}}{2t} = -\dfrac{1}{2t^3}$. At $t = 2$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{16}$.

3. Find the area of the region under the graph of the curve defined by $x(t) = t^2 - 3$, $y(t) = 2t - 1$ between $t = 2$ and $t = 3$.

   Solution. $A = \displaystyle\int_2^3 (2t - 1)(2t)dt = \left(\frac{4}{3}t^3 - t^2\right)\Big|_2^3 = 27 - \frac{20}{3} = \frac{61}{3}$.

Exercises.

1. Set up an integral needed to find the length of the curve defined by $x(t) = t^2 - 3$, $y(t) = 2t - 1$ between $t = 2$ and $t = 3$.

   Solution. $L = \displaystyle\int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_2^3 \sqrt{(2t)^2 + (2)^2}\, dt =$
   $\displaystyle\int_2^3 \sqrt{4t^2 + 4}\, dt = 2\int_2^3 \sqrt{t^2 + 1}\, dt.$

2. Set up an integral needed to find the area of the surface obtained by rotating the curve defined by $x(t) = t^2 - 3$, $y(t) = 2t - 1$ between $t = 2$ and $t = 3$ about the x-axis.

   Solution. $SA = 2\pi \displaystyle\int_2^3 y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$
   $2\pi \displaystyle\int_2^3 (2t - 1)\sqrt{4t^2 + 4}\, dt = 4\pi \int_2^3 (2t - 1)\sqrt{t^2 + 1}\, dt.$

Section 7.3. Polar Coordinates.

Examples.

1. Convert the following polar coordinates into rectangular coordinates: $(3, 0)$, $\left(2, \frac{\pi}{3}\right)$.
2. Convert the following rectangular coordinates into polar coordinates: $(0, 4)$, $(-1, 1)$.
3. Which curves are defined by the following equations in the poloar coordinates?
   (a) $r = 2$
   (b) $\theta = \frac{\pi}{6}$
4. Plot a few points of the curve defined by the function $r = 4 \sin \theta$. Guess the shape of the curve. Then rewrite the above equation in rectangular coordinates and confirm the shape.

Exercises.

1. Plot a few points of the curve defined by the function $r = 1 + \frac{\theta}{\pi}$. Guess the shape of the curve. Then use a graphing calculator to confirm the shape.
2. What do you think the curve will look like for $r = 2 + \frac{\theta}{\pi}$?

   How about $r = \frac{\theta}{\pi}$? Use a graphing calculator to confirm your answers.

Section 7.4. Area and Arc Length in Polar Coordinates.

Examples. Consider the curve defined by $r = 4\sin\theta$. (Recall from a previous example that this is actually a circle of radius 2 centered at $(0,2)$.)

1. Find the length of this curve.

   Solution. The curve is traced once as θ ranges from 0 to π, thus the length is $L = \int_\alpha^\beta \sqrt{(r'(\theta))^2 + (r(\theta))^2}\, d\theta = \int_0^\pi \sqrt{(4\cos\theta)^2 + (4\sin\theta)^2}\, d\theta = \int_0^\pi 4\, d\theta = 4\pi$.

2. Find the area of the region bounded by this curve.

   Solution. Since $4\sin\theta \geq 0$ for all values of θ between 0 and π, the area is $A = \frac{1}{2}\int_0^\pi (r(\theta))^2\, d\theta = \frac{1}{2}\int_0^\pi (4\sin\theta)^2\, d\theta = 8\int_0^\pi \sin^2\theta\, d\theta =$
   $8\int_0^\pi \frac{1}{2}(1 - \cos(2\theta))\, d\theta = 4\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_0^\pi = 4\pi$

Section 7.5. Conic Sections.

Examples.

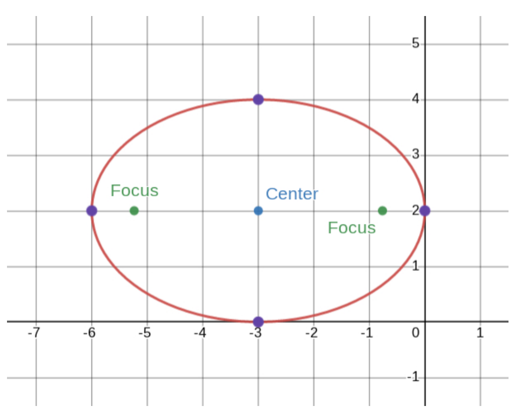
1. The equation $x^2 - 4x - 8y + 12 = 0$ defines a parabola. Write it in the standard form and find the vertex, the focus, and the directrix of the parabola. Sketch the graph.

2. The equation $4x^2 + 24x + 9y^2 - 36y + 36 = 0$ defines an ellipse. Write it in the standard form. Find the center, the foci, and the endpoints of the major and minor axes of the ellipse. Sketch the graph.

   Solution. The equation can be rewritten as follows.

$$\begin{aligned}
(4x^2 + 24x) + (9y^2 - 36y) + 36 &= 0 \\
4(x^2 + 6x) + 9(y^2 - 4y) + 36 &= 0 \\
4(x^2 + 6x + 9) - 36 + 9(y^2 - 4y + 4) - 36 + 36 &= 0 \\
4(x+3)^2 + 9(y-2)^2 - 36 &= 0 \\
4(x+3)^2 + 9(y-2)^2 &= 36 \\
\frac{(x+3)^2}{9} + \frac{(y-2)^2}{4} &= 1 \\
\frac{(x-(-3))^2}{3^2} + \frac{(y-2)^2}{2^2} &= 1
\end{aligned}$$

   Thus the center of the ellipse is at $(-3, 2)$. Since $a = 3$ and $b = 2$, we have $c = \sqrt{a^2 - b^2} = \sqrt{5}$. The foci are at $(-3 + \sqrt{5}, 2)$ and $(-3 - \sqrt{5}, 2)$. The endpoints of the major axis are at $(0, 2)$ and $(-6, 2)$ and the endpoint of the minor axis are at $(-3, 4)$ and $(-3, 0)$. The graph is shown below.



Exercise.

1. The equation $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ defines a hyperbola. Write it in the standard form. Find the vertices and the asymptotes of the hyberbola. Sketch the graph.

   Solution. This is example 7.21 in the book.