Problem Solving Session (aka MFD prep)
CSU Fresno
February 27, 2015
Topics: Composition of functions, Complex numbers

Problem Solving Sessions website:
`http://zimmer.csufresno.edu/~mnogin/mfd-prep.html`

Math Field Day date: Saturday, April 18, 2015

Math Field Day website:
`http://www.fresnostate.edu/csm/math/news-and-events/field-day/`

Mini Mad Hatter (individual, 2 minutes per problem)

1. (MH 2012 9-10) Let $f(x) = x^2 + 6$ and $g(x) = 2x^2$. What is $f(g(x))$?
   (a) $4x^4 + 6$
   (b) $4x^2 + 12$
   (c) $2x^4 + 12x^2$
   (d) $2x^4 + 24x^2 + 72$

   Solution. (a)
   $f(g(x)) = f(2x^2) = (2x^2)^2 + 6 = 4x^4 + 6$

2. (appeared on MFD a few times, modified to the current year) If i is the imaginary number, what is i^{2015}?
   (a) 1
   (b) -1
   (c) i
   (d) $-i$
   (e) None of the above

   Solution. (d)
   $i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$

   $i^{2015} = i^{2012} \cdot i^3 = (i^4)^{503} \cdot i^3 = -i$

3. (MH 2012 11-12) Determine the real part of $(1 + i)^5$.
   (a) 1
   (b) $4\sqrt{2}$
   (c) 21
   (d) 32
   (e) None of the above

   Solution. (e)

$(1+i)^2 = 1 + 2i + (-1) = 2i$
$(1+i)^5 = (1+i)^2(1+i)^2(1+i) = (2i)(2i)(1+i) = -4(1+i) = -4 - 4i.$
The real part is -4.

4. (MH 2011 9-10) If $f(x) = x + 2$ and $f(g(1)) = 6$, which of the following could be $g(x)$?

   (a) $3x$

   (b) $x + 3$

   (c) $x - 3$

   (d) $2x + 1$

   Solution. (b)
   $f(g(1)) = 6$
   $g(1) + 2 = 6$
   $g(1) = 4$
   The only answer choice that satisfies the above condition is $x + 3$.

5. (MH 2010 9-10) Compute $f(g(4))$ if $f(4) = -4$, $g(4) = -2$, and $f(-2) = -1$.

   (a) 8

   (b) 4

   (c) -2

   (d) -1

   Solution. (d)
   $f(g(4)) = f(-2) = -1$

6. (MH 2010 11-12) If $i = \sqrt{-1}$, what is the value of $\left(\frac{1+i}{1-i}\right)^{2010}$?

   (a) 0

   (b) 1

   (c) $-i$

   (d) -1

   (e) i

   Solution. (d)
   $\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+2i+i^2}{1-(-1)} = \frac{2i}{2} = i, \quad \left(\frac{1+i}{1-i}\right)^{2010} = i^{2010} = (i^2)^{1005} = (-1)^{1005} = -1$

7. (MH 2011 11-12) The distance between the two complex numbers $1 - 2i$ and $2i - 2$ is:

   (a) $3 - 4i$

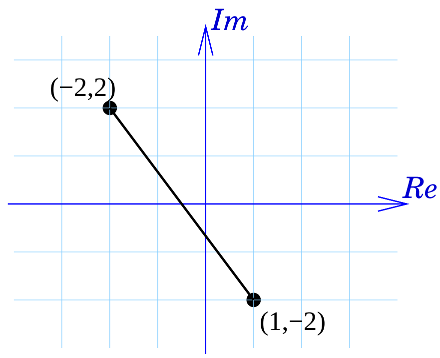
   (b) $-3 + 4i$

   (c) 5

   (d) $\sqrt{17}$

(e) None of the above

Solution. (c)



$d = \sqrt{(-2-1)^2 + (2-(-2))^2} = \sqrt{3^2+4^2} = 5$

8. (MH 2011 11-12) For what values of b is $\frac{2+i}{bi-1}$ a real number? (Here $i^2 = -1$.)
   (a) -1.5
   (b) -0.5
   (c) 0.5
   (d) 2
   (e) 1.5

   Solution. (b)
   $\frac{2+i}{bi-1} = \frac{(2+i)(bi+1)}{(bi-1)(bi+1)} = \frac{2bi+2-b+i}{-b^2-1} = \frac{2-b}{-b^2-1} + \frac{2b+1}{-b^2-1}i$
   The imaginary part is 0 when $2b+1=0$, so $b = -\frac{1}{2}$.

Mini Leap Frog (2 participants per team)

1. (LF 2014 11-12) Given that $2+\sqrt{3}$ is one of the solutions of the equation $x^4 - 14x^3 + 54x^2 - 62x + 13 = 0$, how many complex non-real solutions does this equation have?
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

   Solution. (a)
   Since $2+\sqrt{3}$ is a root, $2-\sqrt{3}$ must be a root also. Then the LHS is divisible by $(x-(2+\sqrt{3}))(x-(2-\sqrt{3})) = ((x-2)-\sqrt{3})((x-2)+\sqrt{3})) = (x-2)^2-3 = x^2-4x+1$. Dividing the LHS by x^2-4x+1 gives $x^2-10x+13$, so the given equation is equivalent to

   $$(x^2-4x+1)(x^2-10x+13) = 0.$$

   Now, the equation $x^2-10x+13=0$ has two real roots (namely, $5 \pm 2\sqrt{3}$), so there are no complex solutions.

2. (MH 2012 11-12) Suppose $f(x) = ax + b$ where a and b are real numbers. We define

$$f_1(x) = f(x)$$

and

$$f_{n+1}(x) = f(f_n(x))$$

for all positive integers n. If $f_7(x) = 128x + 381$, what is the value of $a + b$?

(a) 1

(b) 2

(c) 5

(d) 7

(e) 8

Solution. (c)
$f_1(x) = ax + b$
$f_2(x) = a^2x + ab + b$
$f_3(x) = a^3x + a^2b + ab + b$
$f_4(x) = a^4x + a^3b + a^2b + ab + b$
...
$f_7(x) = a^7x + a^6b + a^5b + a^4b + a^3b + a^2b + ab + b$
$a^7x + (a^6 + a^5 + a^4 + a^3 + a^2 + a + 1)b = 128x + 381$
$a^7 = 128$
$a = 2$
$(64 + 32 + 16 + 8 + 4 + 2 + 1)b = 381$
$127b = 381$
$b = 3$
$a + b = 2 + 3 = 5$

3. (LF 2008 9-12) Let $f(x) = |2x - 3|$. How many real solutions, x, are there to the equation $f(f(x)) = 3$?

(a) 4

(b) 3

(c) 2

(d) 1

(e) None of these

Solution. (b)
$|2|2x - 3| - 3| = 3$
$2|2x - 3| - 3 = \pm 3$
$2|2x - 3| - 3 = 3 \quad \text{or} \quad 2|2x - 3| - 3 = -3$
$2|2x - 3| = 6 \quad \text{or} \quad 2|2x - 3| = 0$
$|2x - 3| = 3 \quad \text{or} \quad |2x - 3| = 0$
$2x - 3 = \pm 3 \quad \text{or} \quad 2x - 3 = 0$
$2x - 3 = 3 \quad \text{or} \quad 2x - 3 = -3 \quad \text{or} \quad 2x - 3 = 0$
$x - 3 \quad \text{or} \quad x = 0 \quad \text{or} \quad x = \frac{3}{2}$

4. (LF 2014 11-12) Let $f(x) = |3x - 2|$. Find the sum of all real solutions, x, to the equation $f(f(x)) = 2$.

   (a) 2

   (b) $\frac{14}{9}$

   (c) $\frac{16}{3}$

   (d) 0

   (e) None of the above

   Solution. (a)
   $|3|3x-2|-2| = 2$
   $3|3x-2|-2 = \pm 2$
   $3|3x-2|-2 = 2 \quad \text{or} \quad 3|3x-2|-2 = -2$
   $3|3x-2| = 4 \quad \text{or} \quad 3|3x-2| = 0$
   $|3x-2| = \frac{4}{3} \quad \text{or} \quad |3x-2| = 0$
   $3x-2 = \frac{4}{3} \quad \text{or} \quad 3x-2 = -\frac{4}{3} \quad \text{or} \quad 3x-2 = 0$
   $x_1 = \frac{10}{9}, \quad x_2 = \frac{2}{9}, \quad x_3 = \frac{2}{3} = \frac{6}{9}$
   $x_1 + x_2 + x_3 = \frac{18}{9} = 2$

5. (LF 2006 9-12) If $i = \sqrt{-1}$, then $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2006} =$

   (a) $\frac{1}{2} - i\frac{\sqrt{3}}{2}$

   (b) $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$

   (c) $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

   (d) $\frac{1}{2} + i\frac{\sqrt{3}}{2}$

   (e) None of these

   Solution. (c)
   $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
   $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1$

   Since $2006 = 668 \cdot 3 + 2$,
   $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2006} = \left(\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3\right)^{668}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = (-1)^{668}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

6. (MH 2011 11-12) Suppose a, b are positive integers, $a < 10$ and $f(x) = ax + b$, $g(x) = bx + a$. If

   $$f(g(50)) - g(f(50)) = 28,$$

   what is (a, b)?

   (a) $(3, 4)$

   (b) $(7, 4)$

   (c) $(6, 2)$

(d) $(4,1)$

(e) $(5,2)$

Solution. (c)
$f(g(x)) = a(bx+a)+b = abx+a^2+b$
$g(f(x)) = b(ax+b)+a = abx+b^2+a$
$f(g(50)) - g(f(50)) = (50ab+a^2+b) - (50ab+b^2+a) = a^2-b^2+b-a$
$a^2-b^2+b-a = 28$
$(a-b)(a+b-1) = 28$
Answer (a) can be eliminated because $a < b$.
Also, (b), (d), and (e) can be eliminated because $a-b=3$ and 3 does not divide 28.
Answer (c) is left, and indeed it works.

More practice problems

1. (MH 2010 11-12) Suppose w is a complex number satisfying $w^2+2w+4=0$. Then $w^6 = ?$

   (a) 1

   (b) 2

   (c) 8

   (d) 32

   (e) 64

   Solution. (e)
   One way is to solve the quadratic equation (e.g. using the quadratic formula) and then compute w^6.
   Another way:
   $w^2 = -2w-4$
   $w^3 = -2w^2-4w = -2(-2w-4)-4w = 4w+8-4w = 8$
   $w^6 = (w^3)^2 = 64$

2. (MH 2011 11-12) Suppose $f(0)=3$ and $f(n)=f(n-1)+2$. Let $T = f(f(f(f(5))))$. What is the sum of the digits of T?

   (a) 6

   (b) 7

   (c) 8

   (d) 9

   (e) 10

   Solution. (c)
   Compute $f(1)$, $f(2)$, $f(3)$, and notice that $f(n)=2n+3$.
   Then $f(f(f(f(5)))) = f(f(f(13))) = f(f(29)) = f(61) = 125$.
   $1+2+5=8$.

3. (MH 2013 11-12) The set $\left\{\frac{z-1}{z+1} \mid z \in \mathbb{C},\ |z|<1\right\}$ is:

(a) a circle

(b) the entire complex plane

(c) the open left half of the complex plane

(d) the open right half of the complex plane

(e) the complex plane except for the real axis

Solution. (c)
One way to solve this problem is to use the process of elimination.
If $z = 0$, $\frac{z-1}{z+1} = -1$, so this value eliminates answers (d) and (e).
The only value of z that makes $\frac{z-1}{z+1} = 0$ is $z = 1$, but we are given $|z| < 1$, so this eliminates answer (b).
Finally, by choosing z sufficiently close to -1, we can make the absolute value of the denominator as close as we want to 0, thus making the value of $\frac{z-1}{z+1} = 0$ as far as we want from the origin. This eliminates answer (a) because any circle is bounded.
Thus only answer (c) remains.
Another way is to rewrite $\frac{z-1}{z+1}$ as $1 - \frac{2}{z+1}$ and use complex plane transformations, but that requires knowledge of some college level complex analysis.

4. (LF 2013 11-12) Find the imaginary parts of the roots of $iz^2 + (2+i)z + 1$.

   (a) $\frac{-1\pm\sqrt{3}}{2}$

   (b) $\frac{-2\pm\sqrt{3}}{2}$

   (c) $\frac{1\pm\sqrt{3}}{2}$

   (d) $\frac{2\pm\sqrt{3}}{2}$

   (e) None of the above

   Solution. (d)
   Hint: use the quadratic formula and simplify.

5. (MH 2011 11-12) Let $a, b \in \mathbb{R}$. A student wrote that the product of $a + i$ and $b - i$ was $a + b + i$, where $i^2 = -1$. If this was correct, then the minimum value of ab is:

   (a) 0

   (b) 1

   (c) 2

   (d) -2

   (e) None of the above

   Solution. (a)
   $(a+i)(b-i) = ab - ai + bi + 1 = (ab+1) + i(b-a)$
   $(ab+1) + i(b-a) = (a+b) + i$
   $ab + 1 = a + b, \quad b - a = 1$
   $b = a + 1$
   $a(a+1) + 1 = a + a + 1$
   $a^2 - a = 0$
   $a(a-1) = 0$

$a = 0$ or $a = 1$
If $a = 0$, then $b = 1$ and $ab = 0$.
If $a = 1$, then $b = 2$ and $ab = 2$.

6. (MH 2014 11-12) Let $f(x) = x^2+10x+20$. For what real values of x is $f(f(f(f(x)))) = 0$?
   (a) $\pm 5^{1/4} - 5$
   (b) $\pm 5^{1/8} - 5$
   (c) $\pm 5^{1/10} - 5$
   (d) $\pm 5^{1/12} - 5$
   (e) $\pm 5^{1/16} - 5$

   Solution. (e)
   Notice that $f(x) = (x + 5)^2 - 5$. Let $x = y - 5$. Then
   $f(x) = y^2 - 5$,
   $f(f(x)) = y^4 - 5$,
   $f(f(f(x))) = y^8 - 5$,
   $f(f(f(f(x)))) = y^16 - 5$.
   Solving $y^16 - 5$ gives $y = \pm 5^{1/16}$, so $x = \pm 5^{1/16} - 5$

7. (MH 2013 11-12) If $i = \sqrt{-1}$ and if $(x + iy)^4 = -16$, then x must be
   (a) either 1 or -1
   (b) either 2 or -2
   (c) either $\sqrt{2}$ or $-\sqrt{2}$
   (d) 0
   (e) either $\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

   Solution. (c)
   $(x + iy)^4 = -16$
   $(x + iy)^2 = \pm 4i$
   $x + iy = \pm\sqrt{2} \pm \sqrt{2}i$

8. (LF 2010 9-10) Let $f(x) = \frac{x}{5} + \frac{5}{x}$. How many real numbers x satisfy the equation $f(f(x)) = f(x)$?
   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) None of these

   Solution. (d)
   Let $f(x) = y$. First solve $f(y) = y$ to find $y = \pm\frac{5}{2}$.
   Then solve $f(x) = \pm\frac{5}{2}$. There are 4 solutions: $\pm\frac{5}{2}$, ± 10.