A short proof of a theorem of S.Eilenberg and J.Moore

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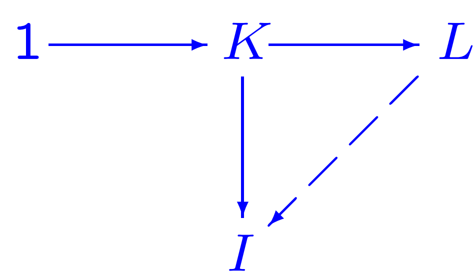
Injective objects in the category of groups

Definition. An object I in a category $\mathfrak{C}$ is called injective if for any monomorphism $K \to L$, and any map $K \to I$, there is a map $L \to I$ such that the diagram



commutes.

Theorem (S. Eilenberg, J. C. Moore) The only injective object in the category of groups is the trivial group.

Lemma (Classical) Let $F[a,b]$ denote the free group on letters a and b. Then the group homomorphism

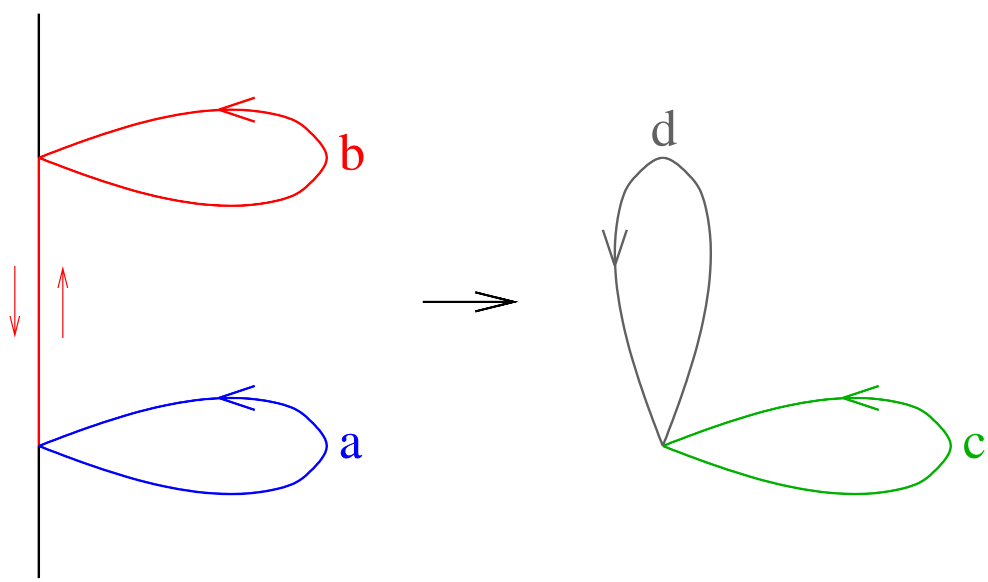
$$F[a,b] \xrightarrow{i} F[c,d]$$

given by

$$\begin{array}{rcl} a & \mapsto & c \\ b & \mapsto & dcd^{-1} \end{array}$$

is an injection.

Proof.



$F[a,b] \hookrightarrow \pi_1$(covering space) $\hookrightarrow \pi_1$(bouquet) $= F[c,d]$

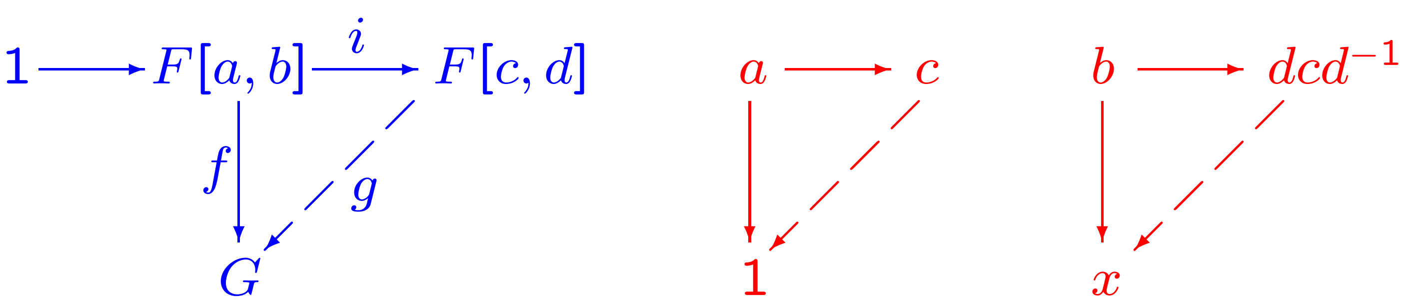
A short proof of theorem

The only injective object in the category of groups is the trivial group.

Proof. Suppose G is injective, let $x \in G$.

$$F[a,b] \xrightarrow{f} G$$

$$\begin{aligned} f(a) &= 1 \\ f(b) &= x \end{aligned}$$



$$g(c) = g(i(a)) = f(a) = 1,$$

and

$$\begin{aligned} x &= f(b) = g(i(b)) = g(dcd^{-1}) = \\ &= g(d)g(c)g(d^{-1}) = g(d)1(g(d))^{-1} = 1. \end{aligned}$$