The Logic of Topology

Maria Nogin
CSU Fresno
mnogin@csufresno.edu

Outline

Preliminaries

Outline

Outline

Outline

- Preliminaries
  - Set operations
  - Classical logic
  - Topological spaces

Outline

Outline

Outline

Outline

Outline

Set operations and logical connectives

Set operations and logical connectives

$$\overline{P \cup Q} = \overline{P} \cap \overline{Q} \qquad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$

Set operations and logical connectives

$$\overline{P \cup Q} = \overline{P} \cap \overline{Q} \qquad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$

$$\overline{\overline{P}} = P \qquad \neg\neg P \equiv P$$

Set operations and logical connectives

$$\overline{P \cup Q} = \overline{P} \cap \overline{Q} \qquad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$

$$\overline{\overline{P}} = P \qquad \neg\neg P \equiv P$$

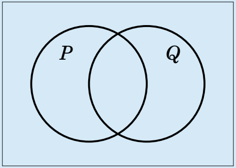
$$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R) \qquad P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

Set operations and logical connectives

$$\overline{P \cup Q} = \overline{P} \cap \overline{Q} \qquad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$

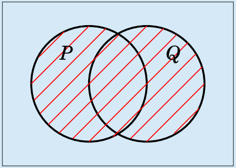
Set operations and logical connectives

$$\overline{P \cup Q} = \overline{P} \cap \overline{Q} \qquad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$



Set operations and logical connectives

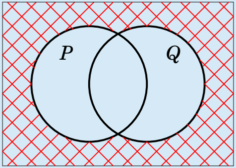
$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$ $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$



$P \cup Q$

Set operations and logical connectives

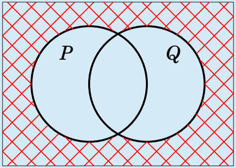
$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$ $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$



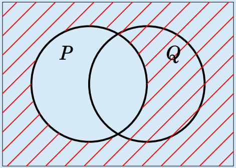
$\overline{P \cup Q}$

Set operations and logical connectives

$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$ $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$



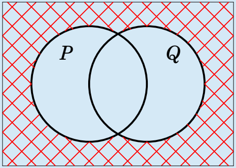
$\overline{P \cup Q}$



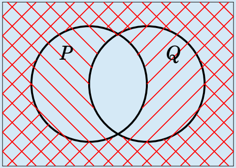
$\overline{P}$

Set operations and logical connectives

$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$ $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$



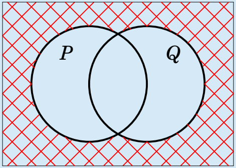
$\overline{P \cup Q}$



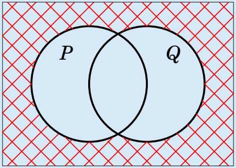
$\overline{P}$, $\overline{Q}$

Set operations and logical connectives

$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$ $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$



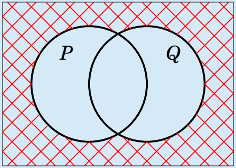
$\overline{P \cup Q}$



$\overline{P} \cap \overline{Q}$

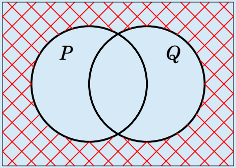
Set operations and logical connectives

$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$ $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$



$\overline{P \cup Q}$

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

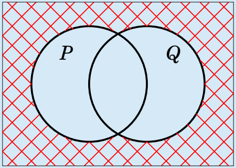


$\overline{P} \cap \overline{Q}$

P	Q	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

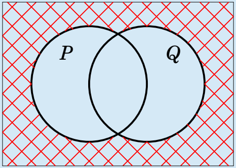
Set operations and logical connectives

$\overline{P \cup Q} = \overline{P} \cap \overline{Q}$ $\qquad$ $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$



$\overline{P \cup Q}$

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T



$\overline{P} \cap \overline{Q}$

P	Q	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Axioms

1. $(P \wedge Q) \to P$
2. $(Q \wedge P) \to P$
3. $P \to (P \vee Q)$
4. $P \to (Q \vee P)$
5. $\neg\neg P \to P$
6. $P \to (Q \to P)$
7. $P \to (Q \to (P \wedge Q))$
8. $\big((P \to Q) \wedge (P \to \neg Q)\big) \to \neg P$
9. $\big((P \to R) \wedge (Q \to R)\big) \to \big((P \vee Q) \to R\big)$
10. $\big((P \to Q) \wedge \big(P \to (Q \to R)\big)\big) \to (P \to R)$

Axioms

1. $(P \wedge Q) \to P$
2. $(Q \wedge P) \to P$
3. $P \to (P \vee Q)$
4. $P \to (Q \vee P)$
5. $\neg\neg P \to P$
6. $P \to (Q \to P)$
7. $P \to (Q \to (P \wedge Q))$
8. $\big((P \to Q) \wedge (P \to \neg Q)\big) \to \neg P$
9. $\big((P \to R) \wedge (Q \to R)\big) \to \big((P \vee Q) \to R\big)$
10. $\big((P \to Q) \wedge \big(P \to (Q \to R)\big)\big) \to (P \to R)$

Rule of inference

$$\frac{P, \;\; P \to Q}{Q}$$

Example: derive $(A \vee B) \to (B \vee A)$

1. Axiom $P \to (P \vee Q)$: $B \to (B \vee A)$
2. Axiom $P \to (Q \vee P)$: $A \to (B \vee A)$
3. Axiom $P \to \big(Q \to (P \wedge Q)\big)$:
   $(A \to B \vee A) \to \Big((B \to B \vee A) \to \big((A \to B \vee A) \wedge (B \to B \vee A)\big)\Big)$
4. Steps 2 and 3: $(B \to B \vee A) \to \big((A \to B \vee A) \wedge (B \to B \vee A)\big)$
5. Steps 1 and 4: $(A \to B \vee A) \wedge (B \to B \vee A)$
6. Axiom $\Big((P \to R) \wedge (Q \to R)\Big) \to \Big((P \vee Q) \to R\Big)$:
   $\Big((A \to B \vee A) \wedge (B \to B \vee A)\Big) \to \Big((A \vee B) \to (B \vee A)\Big)$
7. Steps 5 and 6: $(A \vee B) \to (B \vee A)$

Subset interpretation

Let X be a set.
Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
- disjunction $\vee$ – as union $\cup$
- negation $\neg$ – as complement $\overline{}$

Subset interpretation

Let X be a set.
Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
- disjunction $\vee$ – as union $\cup$
- negation $\neg$ – as complement $\overline{}$

$(P \to Q) \equiv ((\neg P) \vee Q)$

Subset interpretation

Let X be a set.
Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
- disjunction $\vee$ – as union $\cup$
- negation $\neg$ – as complement $\overline{}$

$(P \to Q) \equiv ((\neg P) \vee Q)$, $(P \leftrightarrow Q) \equiv ((P \to Q) \wedge (Q \to P))$

Subset interpretation

Let X be a set.
Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
- disjunction $\vee$ – as union $\cup$
- negation $\neg$ – as complement $\overline{}$

$$(P \to Q) \equiv ((\neg P) \vee Q)\ , \quad (P \leftrightarrow Q) \equiv ((P \to Q) \wedge (Q \to P))$$

Given a mapping from propositional variables (P, Q, etc.) to subsets of X, every formula is mapped to a subset X.

e.g.

$$P \wedge Q \quad \mapsto \quad P \cap Q$$
$$P \vee \neg P \quad \mapsto \quad P \cup \overline{P}$$

Subset interpretation

Let X be a set.
Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
- disjunction $\vee$ – as union $\cup$
- negation $\neg$ – as complement $\overline{}$

$(P \to Q) \equiv ((\neg P) \vee Q)$, $(P \leftrightarrow Q) \equiv ((P \to Q) \wedge (Q \to P))$

Given a mapping from propositional variables (P, Q, etc.) to subsets of X, every formula is mapped to a subset X.

e.g.
$$P \wedge Q \mapsto P \cap Q$$
$$P \vee \neg P \mapsto P \cup \overline{P} = X$$

Subset interpretation

Let X be a set.
Logical connectives are interpreted as operations on subsets of X:

- conjunction $\wedge$ – as intersection $\cap$
- disjunction $\vee$ – as union $\cup$
- negation $\neg$ – as complement $\overline{}$

$(P \to Q) \equiv ((\neg P) \vee Q)$, $(P \leftrightarrow Q) \equiv ((P \to Q) \wedge (Q \to P))$

Given a mapping from propositional variables (P, Q, etc.) to subsets of X, every formula is mapped to a subset X.

e.g.
$$P \wedge Q \mapsto P \cap Q$$
$$P \vee \neg P \mapsto P \cup \overline{P} = X$$

Some formulas are always mapped to the whole set X. They are called valid with respect to interpretation in X.

Soundness and completeness

Theorem. Let X be a set.

1. All tautologies (= derivable formulas) of the classical logic are valid with respect to interpretation in X.

Soundness and completeness

Theorem. Let X be a set.

1. All tautologies (= derivable formulas) of the classical logic are valid with respect to interpretation in X. The classical logic is sound with respect to this interpretation.

Soundness and completeness

Theorem. Let X be a set.

1. All tautologies (= derivable formulas) of the classical logic are valid with respect to interpretation in X. The classical logic is sound with respect to this interpretation.
2. If X is non-empty, the tautologies (= derivable formulas) of the classical logic are the only formulas valid with respect to interpretation in X.

Soundness and completeness

Theorem. Let X be a set.

1. All tautologies (= derivable formulas) of the classical logic are valid with respect to interpretation in X. The classical logic is sound with respect to this interpretation.
2. If X is non-empty, the tautologies (= derivable formulas) of the classical logic are the only formulas valid with respect to interpretation in X. The classical logic is complete with respect to this interpretation.

Soundness and completeness

Theorem. Let X be a set.

1. All tautologies (= derivable formulas) of the classical logic are valid with respect to interpretation in X. The classical logic is sound with respect to this interpretation.
2. If X is non-empty, the tautologies (= derivable formulas) of the classical logic are the only formulas valid with respect to interpretation in X. The classical logic is complete with respect to this interpretation.

The language of classical logic does not distinguish different non-empty sets X.

Topological spaces

Definition. A topological space is a set X together with a collection of subsets of X, called open subsets, satisfying the following axioms:

- The empty subset and X are open.
- The union of any collection of open subsets is also open.
- The intersection of any pair of open subsets is also open.

Topological spaces

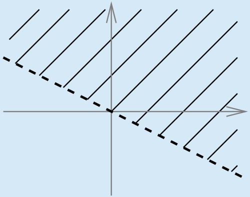
Definition. A topological space is a set X together with a collection of subsets of X, called open subsets, satisfying the following axioms:

- The empty subset and X are open.
- The union of any collection of open subsets is also open.
- The intersection of any pair of open subsets is also open.

Example. $X = \mathbb{R}^n$. A subset P of X is open iff for any point x in P, some open ball containing x is contained in P.

$\mathbb{R}$

$\mathbb{R}^2$



Topological spaces

Definition. The complement of an open subset is called closed.

Topological spaces

Definition. The complement of an open subset is called closed.

Definition. Given a subset P of X, the interior of P is the largest open subset of P.

Example. $X = \mathbb{R}$, $P = [a, b]$, $\text{interior}(P) = (a, b)$.

Topological spaces

Definition. The complement of an open subset is called closed.

Definition. Given a subset P of X, the interior of P is the largest open subset of P.

Example. $X = \mathbb{R}$, $P = [a, b]$, $\text{interior}(P) = (a, b)$.

Definition. Let X and Y be topological spaces. Then $f\colon X \to Y$ is continuous if for any open subset U of Y, $f^{-1}(U)$ is an open subset of X.

Quantifiers

- "$\forall x$" means "for all x"
- "$\exists x$" means "there exists x"

Quantifiers

- "$\forall x$" means "for all x"
- "$\exists x$" means "there exists x"

Example. Let P be a subset of $\mathbb{R}^2$. Then

$$\forall x \in P \; \exists r \in \mathbb{R} \; \Big((r > 0) \wedge \forall y \in \mathbb{R}^2 \big(\mathrm{dist}(x, y) < r \to y \in P \big) \Big)$$

means that P is open.

Quantifiers

- "$\forall x$" means "for all x"
- "$\exists x$" means "there exists x"

Example. Let P be a subset of $\mathbb{R}^2$. Then

$$\forall x \in P\ \exists r \in \mathbb{R}\ \Big((r > 0) \wedge \forall y \in \mathbb{R}^2 \big(\mathrm{dist}(x, y) < r \to y \in P\big)\Big)$$

means that P is open.

The language with quantifiers is very expressive but undecidable.

Compromise: modality

The classical logic is extended with an operator $\Box$.
Interpretations of $\Box$:

- is known
- is provable
- is computable
- is necessary
- will be true tomorrow
- etc.

S4: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\Box$

- Axioms of classical logic
- $\Box P \rightarrow P$
- $\Box P \rightarrow \Box\Box P$
- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

S4: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\Box$

- Axioms of classical logic
- $\Box P \rightarrow P$
- $\Box P \rightarrow \Box\Box P$
- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

Rules of inference

$$\frac{P, \ P \rightarrow Q}{Q} \quad \text{and} \quad \frac{P}{\Box P}$$

S4: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \Box$

- Axioms of classical logic
- $\Box P \rightarrow P$
- $\Box P \rightarrow \Box\Box P$
- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

Rules of inference

$$\frac{P,\ P \rightarrow Q}{Q} \quad \text{and} \quad \frac{P}{\Box P}$$

Topological interpretation of $\Box$:
$\Box P = \text{interior}(P)$

S4: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\Box$

- Axioms of classical logic
- $\Box P \rightarrow P$
- $\Box P \rightarrow \Box\Box P$
- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

Rules of inference

$$\frac{P,\ P \rightarrow Q}{Q} \quad \text{and} \quad \frac{P}{\Box P}$$

Topological interpretation of $\Box$:
$\Box P = \text{interior}(P)$

Theorem. Let X be a topological space. Then S4 is sound with respect to interpretation in X.

Theorem. S4 is complete with respect to all interpretations in all topological spaces X, i.e. for any formula F, the following statements are equivalent:

1. F is derivable in S4
2. F is valid in each interpretation (for each topological space X)

Theorem. S4 is complete with respect to all interpretations in all topological spaces X, i.e. for any formula F, the following statements are equivalent:

1. F is derivable in S4
2. F is valid in each interpretation (for each topological space X)
3. F is valid in each interpretation for each $\mathbb{R}^n$

Theorem. S4 is complete with respect to all interpretations in all topological spaces X, i.e. for any formula F, the following statements are equivalent:

1. F is derivable in S4
2. F is valid in each interpretation (for each topological space X)
3. F is valid in each interpretation for each $\mathbb{R}^n$
4. F is valid in each interpretation for some $\mathbb{R}^n$

Theorem. S4 is complete with respect to all interpretations in all topological spaces X, i.e. for any formula F, the following statements are equivalent:

1. F is derivable in S4
2. F is valid in each interpretation (for each topological space X)
3. F is valid in each interpretation for each $\mathbb{R}^n$
4. F is valid in each interpretation for some $\mathbb{R}^n$

Corollary. The modal logic (with operations $\wedge$, $\vee$, $\neg$, $\to$, $\Box$) does not distinguish R^n's for different n.

Problem

Start with a subset S of $\mathbb{R}$.

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S
$\text{inter}(S)$

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S
inter(S)
compl(inter(S))

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S
inter(S)
compl(inter(S))
inter(compl(inter(S)))

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

$$\begin{array}{l} S \\ \text{inter}(S) \\ \text{compl}(\text{inter}(S)) \\ \text{inter}(\text{compl}(\text{inter}(S))) \\ \vdots \end{array}$$

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S
$\text{inter}(S)$
$\text{compl}(\text{inter}(S))$
$\text{inter}(\text{compl}(\text{inter}(S)))$
$\vdots$

$\text{compl}(S)$

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S
inter(S)
compl(inter(S))
inter(compl(inter(S)))
$\vdots$

compl(S)
inter(compl(S))

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S
$\text{inter}(S)$
$\text{compl}(\text{inter}(S))$
$\text{inter}(\text{compl}(\text{inter}(S)))$
$\vdots$

$\text{compl}(S)$
$\text{inter}(\text{compl}(S))$
$\text{compl}(\text{inter}(\text{compl}(S)))$

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S	compl(S)
inter(S)	inter(compl(S))
compl(inter(S))	compl(inter(compl(S)))
inter(compl(inter(S)))	inter(compl(inter(compl(S))))
$\vdots$	

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

$$\begin{array}{ll} S & \mathrm{compl}(S) \\ \mathrm{inter}(S) & \mathrm{inter}(\mathrm{compl}(S)) \\ \mathrm{compl}(\mathrm{inter}(S)) & \mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S))) \\ \mathrm{inter}(\mathrm{compl}(\mathrm{inter}(S))) & \mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S)))) \\ \vdots & \vdots \end{array}$$

Problem

Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S	compl(S)
inter(S)	inter(compl(S))
compl(inter(S))	compl(inter(compl(S)))
inter(compl(inter(S)))	inter(compl(inter(compl(S))))
⋮	⋮

Can there be infinitely many different sets in these sequences?

Problem

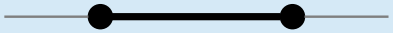
Start with a subset S of $\mathbb{R}$. Consider the following sequences:

S	compl(S)
inter(S)	inter(compl(S))
compl(inter(S))	compl(inter(compl(S)))
inter(compl(inter(S)))	inter(compl(inter(compl(S))))
$\vdots$	$\vdots$

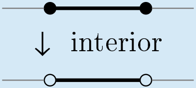
Can there be infinitely many different sets in these sequences?

If not, what is the maximum number of different sets?

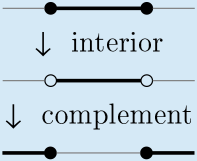
Example 1



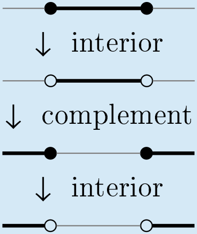
Example 1



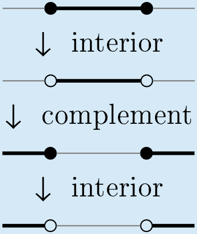
Example 1



Example 1

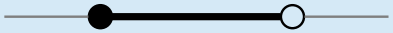


Example 1

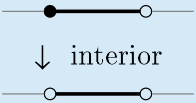


Get 4 different subsets of $\mathbb{R}$

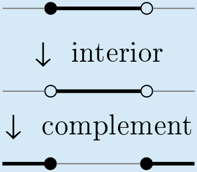
Example 2



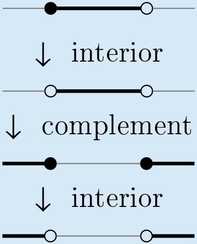
Example 2



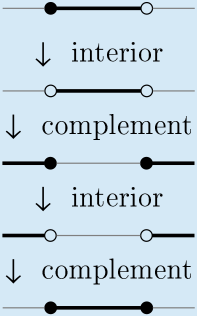
Example 2



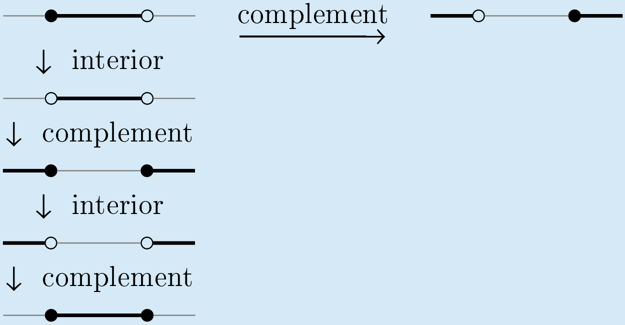
Example 2



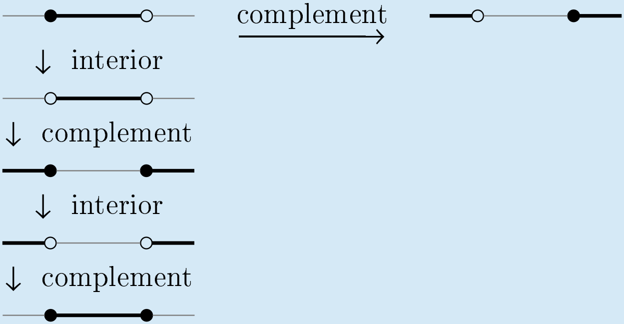
Example 2



Example 2

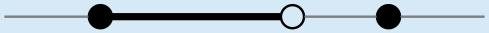


Example 2

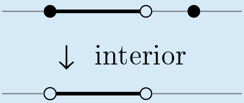


Get 6 different subsets of $\mathbb{R}$

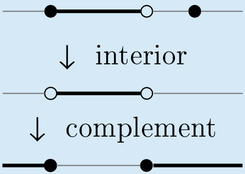
Example 3



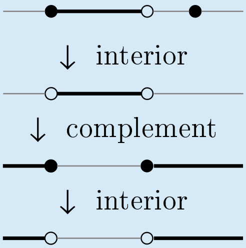
Example 3



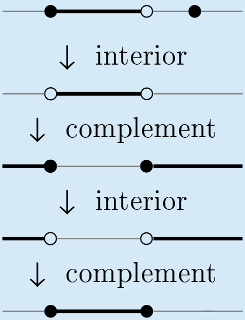
Example 3



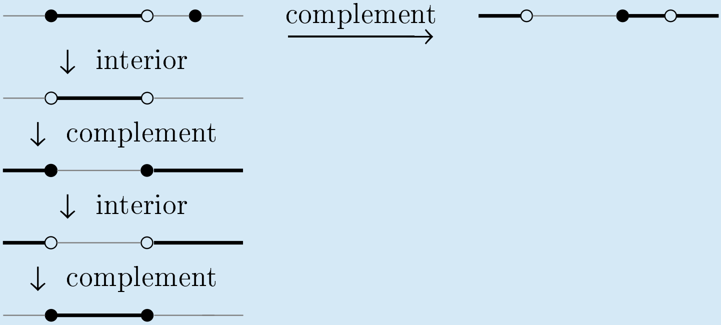
Example 3



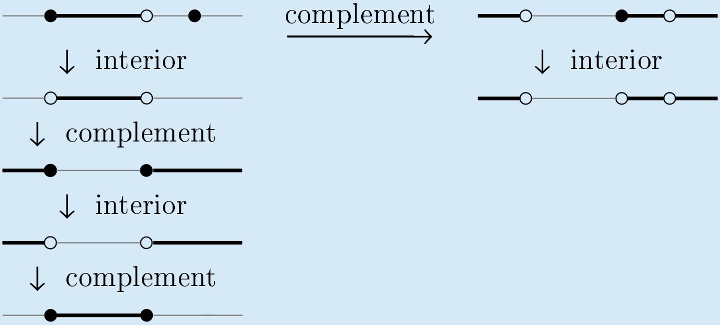
Example 3



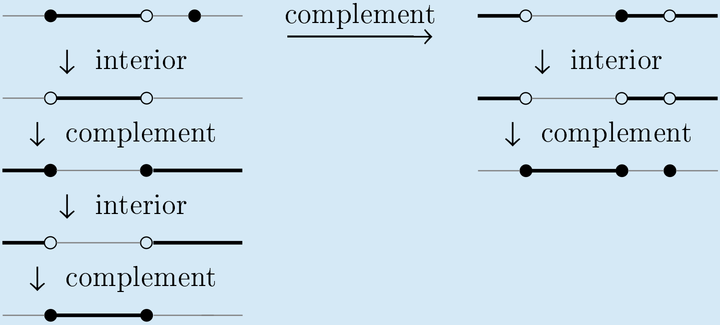
Example 3



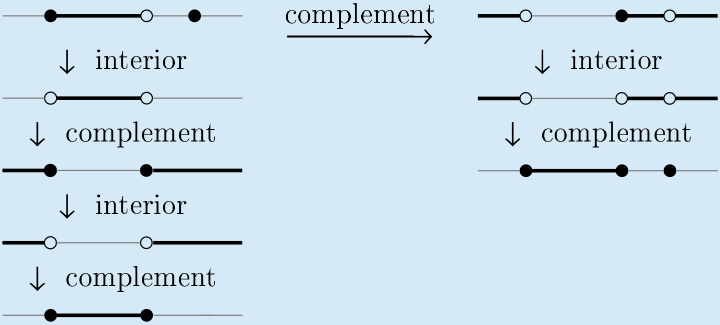
Example 3


complement
interior
complement
interior
complement
interior

Example 3


complement
↓ interior
↓ complement
↓ interior
↓ complement
↓ interior
↓ complement

Example 3



Get 8 different subsets of $\mathbb{R}$

Can there be infinitely many different sets?

Can there be infinitely many different sets?
Answer: No.

Can there be infinitely many different sets?
Answer: No.

What is the largest possible number of different sets?

Can there be infinitely many different sets?
Answer: No.

What is the largest possible number of different sets?
Answer: 14.

Can there be infinitely many different sets?
Answer: No.

What is the largest possible number of different sets?
Answer: 14.

Proof that we cannot get more than 14.

Can there be infinitely many different sets?
Answer: No.

What is the largest possible number of different sets?
Answer: 14.

Proof that we cannot get more than 14.

Lemma. There are at most 7 different sets in the sequence

$$\begin{array}{l} S \\ \mathrm{inter}(S) \\ \mathrm{compl}(\mathrm{inter}(S)) \\ \mathrm{inter}(\mathrm{compl}(\mathrm{inter}(S))) \\ \vdots \end{array}$$

because

$$\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(S))))))) = \mathrm{inter}(\mathrm{compl}(\mathrm{inter}(S))).$$

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\Box\neg\Box\neg\Box\neg\Box\neg T = \Box\neg\Box\neg T$.

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\Box\neg\Box\neg\Box\neg\Box\neg T = \Box\neg\Box\neg T$.

Notation: $\Diamond R \equiv \neg\Box\neg R$.

In the topological interpretation "$\Diamond R$" means "the closure of R".

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\Box\neg\Box\neg\Box\neg\Box\neg T = \Box\neg\Box\neg T$.

Notation: $\Diamond R \equiv \neg\Box\neg R$.

In the topological interpretation "$\Diamond R$" means "the closure of R".

Want to prove: $\Box\Diamond\Box\Diamond T \equiv \Box\Diamond T$.

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\Box\neg\Box\neg\Box\neg\Box\neg T = \Box\neg\Box\neg T$.

Notation: $\Diamond R \equiv \neg\Box\neg R$.
In the topological interpretation "$\Diamond R$" means "the closure of R".

Want to prove: $\Box\Diamond\Box\Diamond T \equiv \Box\Diamond T$.
Proof of $\Box\Diamond T \rightarrow \Box\Diamond\Box\Diamond T$. Axiom: $\Box P \rightarrow P$
Let $P = \neg R$, then $\Box\neg R \rightarrow \neg R$
Contrapositive: $R \rightarrow \neg\Box\neg R$
Let $R = \Box Q$, then $\Box Q \rightarrow \neg\Box\neg\Box Q$
i.e. $\Box Q \rightarrow \Diamond\Box Q$
Apply $\Box$: $\Box\Box Q \rightarrow \Box\Diamond\Box Q$
Axiom: $\Box Q \rightarrow \Box\Box Q$
Therefore $\Box Q \rightarrow \Box\Diamond\Box Q$
Let $Q = \Diamond T$, then $\Box\Diamond T \rightarrow \Box\Diamond\Box\Diamond T$.

Lemma. $\Box\neg\Box\neg\Box\neg\Box S = \Box\neg\Box S$

Proof. Let $T = \neg S$, then $S = \neg T$. We want to prove:
$\Box\neg\Box\neg\Box\neg\Box\neg T = \Box\neg\Box\neg T$.

Notation: $\Diamond R \equiv \neg\Box\neg R$.
In the topological interpretation "$\Diamond R$" means "the closure of R".

Want to prove: $\Box\Diamond\Box\Diamond T \equiv \Box\Diamond T$.
Proof of $\Box\Diamond T \to \Box\Diamond\Box\Diamond T$. Axiom: $\Box P \to P$
Let $P = \neg R$, then $\Box\neg R \to \neg R$
Contrapositive: $R \to \neg\Box\neg R$
Let $R = \Box Q$, then $\Box Q \to \neg\Box\neg\Box Q$
i.e. $\Box Q \to \Diamond\Box Q$
Apply $\Box$: $\Box\Box Q \to \Box\Diamond\Box Q$
Axiom: $\Box Q \to \Box\Box Q$
Therefore $\Box Q \to \Box\Diamond\Box Q$
Let $Q = \Diamond T$, then $\Box\Diamond T \to \Box\Diamond\Box\Diamond T$.
Similarly $\Box\Diamond\Box\Diamond T \to \Box\Diamond T$.

Similarly, there are at most 7 different subsets in the sequence

$$\begin{array}{l} \mathrm{compl}(S) \\ \mathrm{inter}(\mathrm{compl}(S)) \\ \mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S))) \\ \mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S)))) \\ \vdots \end{array}$$

because

$$\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S)))))))) = \mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S)))),$$

so at most 14 different subsets total.

Similarly, there are at most 7 different subsets in the sequence

$\mathrm{compl}(S)$
$\mathrm{inter}(\mathrm{compl}(S))$
$\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S)))$
$\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S))))$
$\vdots$

because

$$\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S)))))))) = \mathrm{inter}(\mathrm{compl}(\mathrm{inter}(\mathrm{compl}(S)))),$$

so at most 14 different subsets total.

Homework problem. Find a subset of $\mathbb{R}$ for which you get 14 different subsets.

Dynamic topological systems

Definition. A dynamic topological system is a topological space X with a continuous function $f\colon X \to X$.

Dynamic topological systems

Definition. A dynamic topological system is a topological space X with a continuous function $f\colon X \to X$.

New modal operator [a]: [a] P is interpreted as $f^{-1}(P)$.

Dynamic topological systems

Definition. A dynamic topological system is a topological space X with a continuous function $f\colon X \to X$.
New modal operator [a]: $[a]\,P$ is interpreted as $f^{-1}(P)$.

S4C

- Axioms of classical logic
- $\Box P \to P$
- $\Box P \to \Box\Box P$
- $\Box(P \to Q) \to (\Box P \to \Box Q)$
- $[a]\,(P \to Q) \to ([a]\,P \to [a]\,Q)$
- $([a]\,\neg P) \leftrightarrow (\neg\,[a]\,P)$
- $([a]\,\Box P) \leftrightarrow (\Box\,[a]\,\Box P)$

Rules of inference

(1) $\dfrac{P,\ P \to Q}{Q}$

(2) $\dfrac{P}{\Box P}$ (3) $\dfrac{P}{[a]\,P}$

Theorem. Let F be a formula. The following are equivalent:

1. F is derivable in S4C
2. F is valid with respect to every interpretation in every $\mathbb{R}^n$

Theorem. Let F be a formula. The following are equivalent:

1. F is derivable in S4C
2. F is valid with respect to every interpretation in every $\mathbb{R}^n$

However, the above statements are not equivalent to

3. F is valid with respect to every interpretation in $\mathbb{R}$

Theorem. Let F be a formula. The following are equivalent:

1. F is derivable in S4C
2. F is valid with respect to every interpretation in every $\mathbb{R}^n$

However, the above statements are not equivalent to

3. F is valid with respect to every interpretation in $\mathbb{R}$

Corollary. The language of S4C distinguishes $\mathbb{R}$ from $\mathbb{R}^n$ for $n > 1$.

Example

Let $U = \Box P$ (U is open),
$\Phi = (\Diamond U) \wedge (\Diamond \neg U)$ (Φ is the boundary of U),

Example

Let $U = \Box P \quad (U \text{ is open}),$
$\Phi = (\Diamond U) \wedge (\Diamond \neg U) \quad (\Phi \text{ is the boundary of } U),$
$\Psi = (\Box\, [\mathrm{a}]\, \Phi) \wedge ([\mathrm{a}]\, Q) \wedge (\Diamond\, [\mathrm{a}]\, \neg Q).$

Example

Let $U = \Box P$ (U is open),
$\Phi = (\Diamond U) \wedge (\Diamond \neg U)$ (Φ is the boundary of U),
$\Psi = (\Box\, [\mathrm{a}]\, \Phi) \wedge ([\mathrm{a}]\, Q) \wedge (\Diamond\, [\mathrm{a}]\, \neg Q)$.

Lemma. If P and Q are subsets of $\mathbb{R}$, then $\Psi = \emptyset$.

Example

Let $U = \Box P \quad (U \text{ is open}),$
$\Phi = (\Diamond U) \wedge (\Diamond \neg U) \quad (\Phi \text{ is the boundary of } U),$
$\Psi = (\Box\, [\mathrm{a}]\, \Phi) \wedge ([\mathrm{a}]\, Q) \wedge (\Diamond\, [\mathrm{a}] \neg Q).$

Lemma. If P and Q are subsets of $\mathbb{R}$, then $\Psi = \emptyset$.

Corollary. $\neg\Psi = \mathbb{R}$

Example

Let $U = \Box P$ (U is open),
$\Phi = (\Diamond U) \wedge (\Diamond \neg U)$ (Φ is the boundary of U),
$\Psi = (\Box\,[\mathrm{a}]\,\Phi) \wedge ([\mathrm{a}]\,Q) \wedge (\Diamond\,[\mathrm{a}]\,\neg Q)$.

Lemma. If P and Q are subsets of $\mathbb{R}$, then $\Psi = \emptyset$.

Corollary. $\neg\Psi = \mathbb{R}$

Lemma. There exist subsets P and Q of $\mathbb{R}^2$ and a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi \neq \emptyset$, i.e. $\neg\Psi \neq \mathbb{R}^2$.

Example

Let $U = \Box P$ (U is open),
$\Phi = (\Diamond U) \wedge (\Diamond \neg U)$ (Φ is the boundary of U),
$\Psi = (\Box\,[\mathrm{a}]\,\Phi) \wedge ([\mathrm{a}]\,Q) \wedge (\Diamond\,[\mathrm{a}]\,\neg Q)$.

Lemma. If P and Q are subsets of $\mathbb{R}$, then $\Psi = \emptyset$.

Corollary. $\neg\Psi = \mathbb{R}$

Lemma. There exist subsets P and Q of $\mathbb{R}^2$ and a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi \neq \emptyset$, i.e. $\neg\Psi \neq \mathbb{R}^2$.

Corollary. The formula $\neg\Psi$ is not derivable in S4C.

Theorem

(joint work with Aleksey Nogin)

For any $n \geq 2$, S4C is complete with respect to topological interpretations in $\mathbb{R}^n$.

Theorem

(joint work with Aleksey Nogin)

For any $n \geq 2$, S4C is complete with respect to topological interpretations in $\mathbb{R}^n$.

Open question

What is the dynamic topological logic of $\mathbb{R}$?

Application: Hybrid Control Systems

- “Discrete” parameters: Discrete Mathematics
- “Continuous” parameters: Optimal Control Theory: Differential Equations, PDEs, etc
- Parameters of both types: Hybrid Control System: Modal Logic

Application: Hybrid Control Systems

- "Discrete" parameters: Discrete Mathematics
- "Continuous" parameters: Optimal Control Theory: Differential Equations, PDEs, etc
- Parameters of both types: Hybrid Control System: Modal Logic

Application: Hybrid Control Systems

- "Discrete" parameters: Discrete Mathematics
- "Continuous" parameters: Optimal Control Theory: Differential Equations, PDEs, etc
- Parameters of both types: Hybrid Control System: Modal Logic

Application: Hybrid Control Systems

- "Discrete" parameters: Discrete Mathematics
- "Continuous" parameters: Optimal Control Theory: Differential Equations, PDEs, etc
- Parameters of both types: Hybrid Control System: Modal Logic

Thank you!