Sampling

Modules: Utilities, Twin Pulse Generator, Digital Utilities, Multiplier, Tuneable LPF, Integrate & Dump, Adder, Wideband True RMS Meter, FM Utilities, Noise Generator

0 Pre-Laboratory Reading

0.1 Ideal Sampling

Ideal sampling produces a train of Dirac delta functions, and the weight of each delta function is proportional to the value of the sampler input at the time of sampling. Dirac delta functions cannot, of course, be realized in practice. Nonetheless, ideal sampling involves relatively simple mathematics and as a result is useful for thought experiments and as a point of departure for characterizing some more practical sampling schemes.

It is important to know the Fourier transform of the sampler output. A train of equal-weight Dirac delta functions (a Dirac comb) in the time domain is a Dirac comb in the frequency domain.
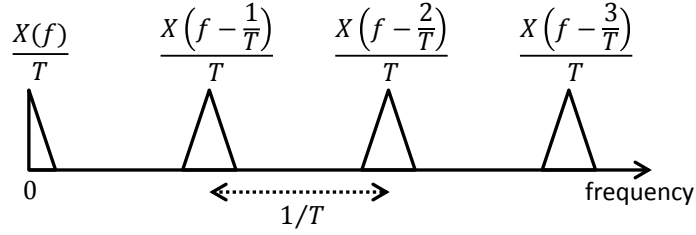
$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) \tag{1}$$
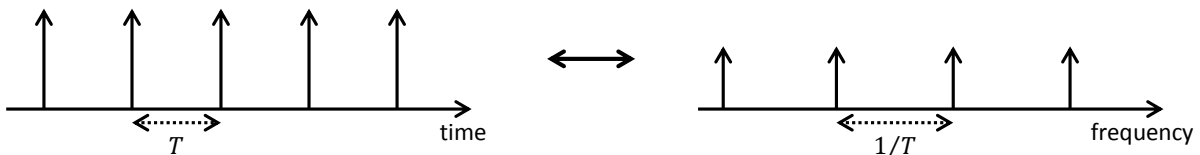
If the spacing between deltas in the time domain is T seconds, then the spacing in the frequency domain is $1/T$ Hz. The output of an ideal sampler may be written as the product of the analog input $x(t)$ and a train of equal-weight deltas. By the Modulation Theorem, the Fourier transform of the product is the convolution (in the frequency domain) of the individual Fourier transforms. With $x(t) \leftrightarrow X(f)$ and since the convolution of $X(f)$ with $\delta(f - k/T)$ is $X(f - k/T)$,

$$x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T}\right) \tag{2}$$

The output of an ideal sampler, when viewed in the frequency domain, is a set of equally spaced (scaled) copies of $X(f)$. The spacing between copies is $1/T$ Hz, and all copies have the same scaling factor $(1/T)$.

The spectral diagram shows copies labeled $\frac{X(f)}{T}$, $\frac{X\left(f-\frac{1}{T}\right)}{T}$, $\frac{X\left(f-\frac{2}{T}\right)}{T}$, $\frac{X\left(f-\frac{3}{T}\right)}{T}$ spaced by $1/T$, starting at 0, along the frequency axis.

The analog signal $x(t)$ can be recovered from the sampler output with a low-pass filter, which is called a reconstruction filter, having a bandwidth of approximately $1/(2T)$ Hz. This filter passes the baseline spectral copy (the one centered at $f = 0$) and rejects the others.

0.2 Natural Sampling

Natural sampling is a practical method that approximates ideal sampling. The analog input $x(t)$ is multiplied by a train of uniformly spaced, rectangular pulses. The figure below illustrates the difference between ideal and natural sampling. If the width T_p of the pulses is much smaller than the spacing T between pulses, then natural sampling may be regarded as an approximation of ideal sampling.

Ideal Sampling   Natural Sampling
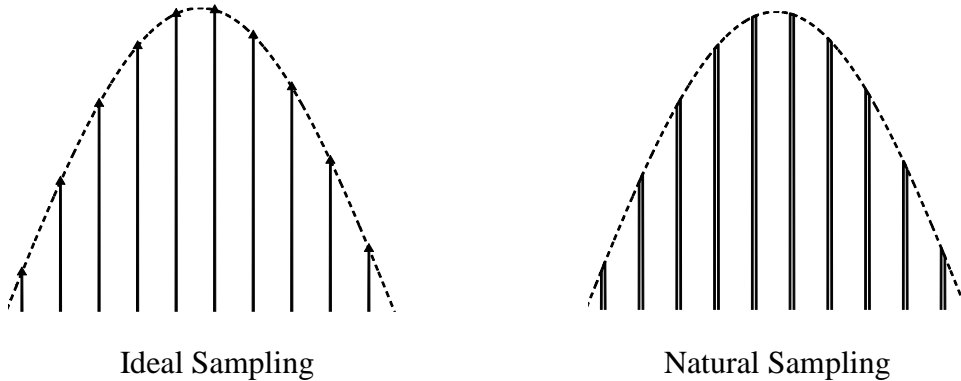
Denoting an individual rectangular pulse by $p(t)$, a train of such pulses, uniformly spaced in time by T seconds, may be expressed as
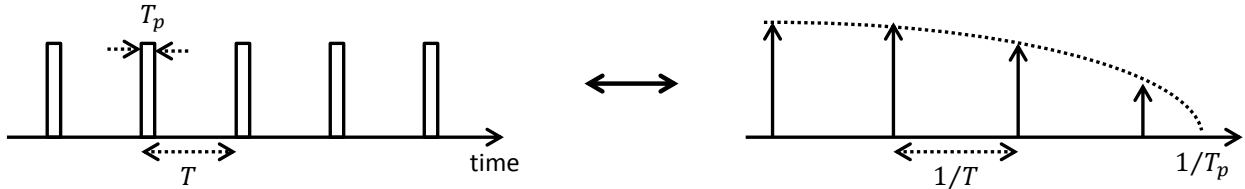
$$\sum_{n=-\infty}^{\infty} p(t - nT) = p(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{3}$$

If the Fourier transform of $p(t)$ is $P(f)$, then the Fourier transform of the pulse train is

$$P(f) \cdot \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) = \sum_{k=-\infty}^{\infty} \frac{1}{T} P\left(\frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right) \tag{4}$$

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Therefore,

$$\sum_{n=-\infty}^{\infty} p(t - nT) \leftrightarrow \sum_{k=-\infty}^{\infty} \frac{1}{T} P\left(\frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right) \tag{5}$$


When this pulse train and the analog signal $x(t)$ are multiplied, the result of this natural sampling is

$$x(t) \cdot \sum_{n=-\infty}^{\infty} p(t - nT) \leftrightarrow X(f) * \sum_{k=-\infty}^{\infty} \frac{1}{T} P\left(\frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right) \tag{6}$$

which simplifies to

$$x(t) \cdot \sum_{n=-\infty}^{\infty} p(t - nT) \leftrightarrow \sum_{k=-\infty}^{\infty} \frac{1}{T} P\left(\frac{k}{T}\right) X\left(f - \frac{k}{T}\right) \tag{7}$$

As in the case of ideal sampling, the spectrum contains uniformly spaced (scaled) copies of $X(f)$, with a spacing of $1/T$ Hz. In the case of natural sampling, however, the spectral copies have different scaling factors: $P(k/T)/T$.
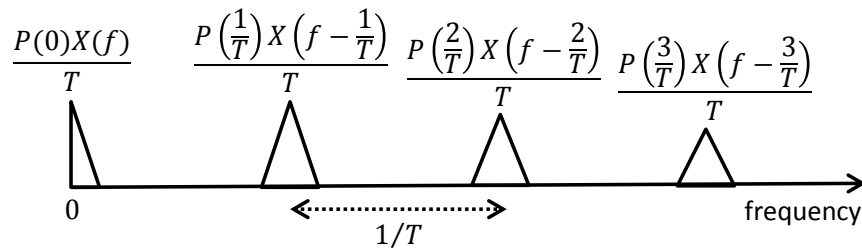
The magnitude of the Fourier transform of a rectangular pulse equals the absolute value of a sinc function. The first null of this $P(f)$ occurs at $f = 1/T_p$ Hz, where T_p is the width of the pulse. For $0 \leq f \leq \frac{0.4}{T_p}$, the spectrum of natural sampling is approximately the same as for ideal sampling. However, for $f > \frac{0.4}{T_p}$ the scaled spectral copies become smaller.

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As with ideal sampling, the analog signal $x(t)$ can be recovered from the sampler output with a reconstruction filter having a bandwidth of approximately $1/(2T)$ Hz.

0.2 Sample and Hold

The most common sampler is the sample-and-hold device. The figure below illustrates how this device works. The analog signal $x(t)$ is sampled every T seconds. The sampler output holds steady at the last sample value, until a new sample is available. In this way, the sampler output is a staircase approximation to $x(t)$.
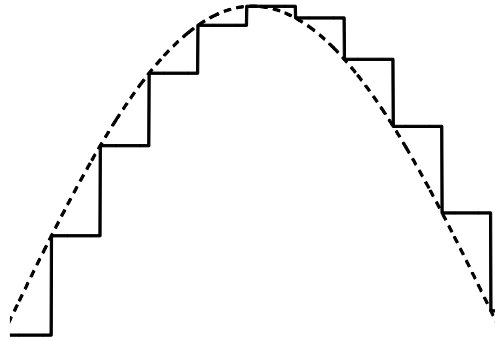
Sample and Hold

A sample and hold is closely related to ideal sampling:

$$p(t) * \left[x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \right] \tag{8}$$

With sample and hold, the pulse shape $p(t)$ has a width equal to T, the sampling period.

$$p(t) * \left[x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \right] \leftrightarrow \frac{1}{T} P(f) \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T} \right) \tag{9}$$

$P(f)$ has a magnitude equal to the absolute value of a sinc function. For the case of a sample-and-hold device, the first null occurs at $f = 1/T$ Hz. If the bandwidth of $x(t)$ is much less than $1/T$ Hz, then the baseline spectral copy (the one centered at $f = 0$) is relatively undistorted. All other spectral copies are badly distorted; however, this is generally not a problem since the analog signal is recovered from the baseline spectral copy.

Sample and hold is more common than natural sampling. The reason for this is that the process of analog-to-digital conversion does not end with sampling. In order to get a digital signal, one that is discrete in amplitude as well as discrete in time, the samples must be quantized. This is accomplished more easily if each sample is held for an entire clock period.

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1 Natural Sampling

You will generate an analog signal that will be used throughout this experiment as the signal that is to be sampled. Place the 2-kHz analog signal (Master Signals) on the input of the rectifier (Utilities module). Observe the rectifier output on the oscilloscope.

      Channel A: rectifier output

This is half-wave rectification. Since your signal is periodic, it possesses a Fourier series expansion. In other words, the frequency content of this signal lies entirely within a discrete set of frequencies (as opposed to occupying a continuum of frequencies). There is a DC term (since the average is non-zero). Then there are the harmonics, the first of which is the fundamental, having a frequency of (100/48) kHz, approximately 2,083.3 Hz. The second harmonic has a frequency of (100/24) kHz, approximately 4,166.7 Hz. The third harmonic has a frequency of (100/16) kHz, which is 6,250 Hz. (Since the characteristic curve of a rectifier has neither even nor odd symmetry, all harmonics—even and odd—are present in the output.)

      Channel A: rectifier output

You will generate a train of narrow pulses that will multiply your analog signal. For this purpose you need the Twin Pulse Generator. Before inserting this module into the TIMS instrument frame, look at the PCB. You will see a slide switch; set it to *single* mode.

In the first instance, you will supply a 50-kHz TTL clock to the Twin Pulse Generator. You can create a 50-kHz clock by connecting the Master Signals 100-kHz TTL clock to a Divide-by-2 (TTL) input of the Digital Utilities module. The (TTL) output of this Divide-by-2 is a 50-kHz clock. Every low-to-high transition occurring on this clock will generate a narrow pulse on the TTL Q_1 output of the Twin Pulse Generator. You can adjust the pulse width using the width knob. Rotate the width knob fully counter-clockwise in order to produce the narrowest possible pulse. Observe the pulse train on the oscilloscope. Use the 50-kHz clock as the external trigger source (and, as always with a TTL signal, set the trigger level to a nonzero, positive value.)

      Channel A: TTL Q_1 output of Twin Pulse Generator

Measure the pulse width using a time ruler and record it. This pulse width will be denoted T_p in this laboratory.

Now set up the multiplication. Connect the narrow-pulse train (the TTL Q_1 signal) to one input of the Multiplier. Connect your analog signal (the rectifier output) to the other input of the Multiplier. Make sure the switch on the front panel of the Multiplier module is set to DC. Connect the analog signal (the rectifier output) to Channel A and the output of the Multiplier to

Channel B. Input coupling for the oscilloscope should be DC. Use a TTL signal with frequency (100/48) kHz as the external trigger source. This trigger source can be obtained from the (100/12) kHz (8.3-kHz) TTL clock using a divide-by-four device.

Channel A: rectifier output
Channel B: Multiplier output (50-kHz sampling)

The Multiplier output should look like periodic narrow pulses, the height of each pulse being proportional to the current value of the analog signal. This is natural sampling.

Observe the spectrum of the Multiplier output (the narrow-pulse train).

Channel B: Multiplier output (50-kHz sampling)

The spectrum should consist of uniformly spaced copies of the original spectrum (that of the rectifier output), where the spacing between adjacent copies equals the sampling rate. Because this natural sampling is only an approximation to ideal sampling, the spectral copies will decrease in magnitude for frequencies f greater than about $0.4/T_p$.

Measure the line height (in dBu) of the upper fundamental in each of several spectral copies. You will want to use a signal ruler for this purpose. You will record the results in an Excel worksheet. A couple of examples might make this clearer. For the baseline spectral copy (centered at 0 Hz), the frequency of the upper fundamental is approximately 2 kHz. For the spectral copy that is centered at 50 kHz, the frequency of the upper fundamental is approximately 52 kHz.

natural sampling

Now you will demonstrate that an approximation to the original analog signal can be recovered from the sampled signal (the Multiplier output). You will use a Tuneable LPF, serving in the role of a reconstruction filter, for this purpose. With the Tuneable LPF set for wide mode, adjust the bandwidth of this filter to approximately 25 kHz. (The Tuneable LPF's clock output has a frequency equal to 100 times the bandwidth.) Use the Noise Generator module to get a quick display of $|H(f)|$.

Channel A: Tuneable LPF output

Connect the Multiplier output to the input of the Tuneable LPF. You will use the Tuneable LPF, set for a bandwidth of 25 kHz, as a reconstruction filter.

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natural sampling

analog signal → X → reconstruction filter →

pulse train
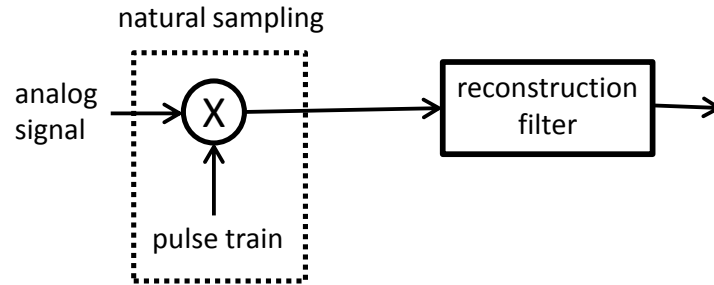
You will normally want the reconstruction filter bandwidth set to approximately one-half of the sampling rate. The filter will pass the baseline spectral copy. The other spectral copies will be rejected. You don't want the filter bandwidth to be too small. If the filter bandwidth is much less than one-half the sampling rate, then part of the baseline spectral copy might get clobbered.

If the filter bandwidth is set equal to one-half of the sampling rate but this bandwidth is too narrow to pass without distortion the baseline spectral copy, then your sampling rate is too small for this particular analog signal. In other words, an inadequate sampling rate causes aliasing, and it then becomes impossible to recover the original analog signal from the samples.

Connect the original analog signal (the rectifier output) to Channel A. Connect the reconstruction filter output to Channel B.

Channel A: rectifier output
Channel B: output of reconstruction filter (25-kHz bandwidth for 50-kHz sampling)

You should find that the reconstruction filter produces a good approximation of the original analog signal. The recovered signal will be delayed somewhat relative to the original; this is to be expected since the reconstruction filter, like all filters, has delay. The recovered signal will also be scaled in amplitude relative to the original. That's okay; if the output of the reconstruction filter has approximately the same form as the original, with perhaps some amplitude scaling and with some delay, we consider that the signal has been successfully recovered.

Now reduce the bandwidth of the reconstruction filter to approximately 3 kHz. You should find that this filter only passes the DC term and the fundamental harmonic. This demonstrates that the 3-kHz bandwidth is inadequate.

Channel A: rectifier output
Channel B: output of reconstruction filter (3-kHz bandwidth for 50-kHz sampling)

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Change the sampling rate. Use 25 kHz, instead of 50 kHz. You can get a 25-kHz clock by using a Divide-by-4, rather than a Divide-by-2, in Digital Utilities. This 25-kHz clock will replace the 50-kHz clock at the TTL input of the Twin Pulse Generator. Connect the analog signal (the rectifier output) to Channel A and the Multiplier output to Channel B. Input coupling for the oscilloscope should be DC. Use a TTL signal with frequency (100/48) kHz as the external trigger source.

> Channel A: rectifier output
> Channel B: Multiplier output (25-kHz sampling)

Observe the spectrum of this Multiplier output.

> Channel B: Multiplier output (25-kHz sampling)

In the frequency domain the spacing between copies of the original analog spectrum is now only half of what it was before. Now the reconstruction filter will need to have a bandwidth of about 12.5 kHz. Set the reconstruction filter bandwidth to approximately 12.5 kHz.

Connect the analog signal (the rectifier output) to Channel A and the reconstruction filter output to Channel B.

> Channel A: rectifier output
> Channel B: output of reconstruction filter (12.5-kHz bandwidth for 25-kHz sampling)

You should find that you have recovered a reasonable approximation to the original analog signal; however, this approximation is not quite as good as the one you obtained with the higher sampling rate.

Keep the sampling rate at 25 kHz and increase the reconstruction filter bandwidth to approximately 25 kHz. You can guess where we are going with this. You are now using a reconstruction filter bandwidth that is too large for the sampling rate. In addition to passing the baseline spectral copy, this filter will also pass parts of the spectral copies located at ±25 kHz. This is bad, but you need to do this once so that you know what bad looks like.

> Channel A: rectifier output
> Channel B: output of reconstruction filter (25-kHz bandwidth for 25-kHz sampling)
> This is nuts!

2 Sample and Hold

A second practical technique for sampling an analog signal is to employ a sample-and-hold circuit. You may remove the Twin Pulse Generator and Multiplier modules. They will not be used in the remainder of this lab.

You will insert an Integrate & Dump module, in order to take advantage of its sample-and-hold circuit. Before inserting the Integrate & Dump module in the TIMS instrument frame, you will need to set a rotary switch on this module's PCB. There are actually two rotary switches on this PCB. Set one of these to S&H (that is, sample-and-hold). If you set rotary switch 1 to S&H, then you will use analog channel 1 on the front panel.

For the remainder of this lab, use a sampling rate of 50 kHz. Connect a 50 kHz TTL clock to the TTL clock input of the Integrate & Dump module. Connect the analog signal (the rectifier output) to the sample-and-hold input. Connect the original analog signal (the rectifier output) to Channel A and the sample-and-hold output to Channel B.

> Channel A: rectifier output
> Channel B: sample-and-hold output

The sample-and-hold output should look like a staircase approximation to the original analog signal.

Observe the spectrum of the sample-and-hold output.

> Channel B: sample-and-hold output

The spectrum of a sample-and-hold output will consist of uniformly-spaced copies of the original analog spectrum (where the spacing between adjacent copies equals the sampling rate), but the magnitude of the copies decrease as the frequency increases. The baseline spectral copy should be a reasonably good replica of the original analog spectrum.

Measure the line height (in dBu) of the upper fundamental in each of several spectral copies. You will want to use a signal ruler for this purpose. You will record the results in an Excel worksheet.

> sample and hold

You will use a Tuneable LPF as the reconstruction filter. With the Tuneable LPF set for wide mode, adjust the bandwidth of this filter to approximately 25 kHz. Connect the output of the sample-and-hold device to the input of the Tuneable LPF.
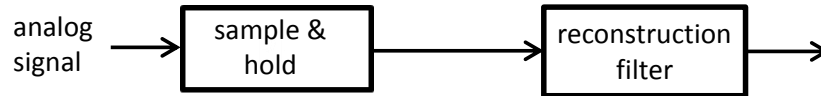
Connect the original analog signal (the rectifier output) to Channel A and the reconstruction filter output to Channel B.

Channel A: rectifier output
Channel B: output of reconstruction filter (25-kHz bandwidth for 50-kHz sampling)

You should find that the output of the reconstruction filter gives a good approximation to the original analog signal (but with delay and amplitude scaling).

Plot the measured line height (in dBu) of the upper fundamental in each of several spectral copies for *natural sampling* with a 50-kHz sampling rate, using discrete points. Also plot (on the same set of axes) the measured line height (in dBu) of the upper fundamental in each of several spectral copies for *sample-and-hold* with a 50-kHz sampling rate, using discrete points.

3 Anti-Aliasing Filter

You will now perform an experiment in which there is a desired signal and an interfering (that is, unwanted) signal. Both signals are sinusoids. For the desired signal, use the 2-kHz analog signal from the Master Signals panel. To generate the interfering signal, you will need the FM Utilities module. At the top of that module is an analog frequency divider. Place a 100-kHz analog sinusoid at the input of this divider, and the output is then a (100/9)-kHz analog sinusoid, called the 11.1-kHz sinusoid.

Combine the desired 2-kHz sinusoid and the interfering 11.1-kHz sinusoid using the Adder. Use the RMS Meter and adjust the gain knobs on the Adder. With only the 2-kHz sinusoid present on the Adder input, the Adder output should be 1 V rms. With only the 11.1-kHz sinusoid present on the input, the output should be 1 V rms.

Place the Adder output on Channel A. Place a copy of the 2-kHz sinusoid on the input of a Buffer Amplifier. Use the RMS Meter to adjust the amplifier gain so that the amplifier output is 1 V rms. Place the amplifier output on Channel B.

In order to stabilize the oscilloscope display, you should use an external trigger source with a frequency of (100/144) kHz. Note that the desired frequency, (100/48) kHz, is 3 times that of the trigger source and the interfering frequency, (100/9) kHz, is 16 times that of the trigger source. You can create the trigger source from the (100/12)-kHz TTL clock (called 8.3 kHz on the

Master Signals panel) using a divide-by-4 device cascaded with a divide-by-3 device (both on the Digital Utilities module).

∿ Channel A: Adder output
Channel B: 2-kHz sinusoid at output of Buffer Amplifier

You should be able to see that the Adder output is the sum of the 2-kHz desired sinusoid with the higher-frequency interfering signal. (Both the Adder and the Buffer Amplifier invert the signal, so the component of the 2-kHz sinusoid on the Adder output should be in phase with the 2-kHz sinusoid on the Buffer Amplifier output.)

With the Adder output still connected to Channel A, note the two significant frequencies that are present.

ᶟ Channel A: Adder output

Place a (100/12)-kHz TTL clock (the 8.3 kHz clock) at the clock input of the Integrate & Dump module. Place the Adder output at the analog input of a sample-and-hold channel. The Adder output should be on Channel A. Place the output of the sample-and-hold channel on Channel B. As above, you should use a (100/144)-kHz TTL external trigger.

∿ Channel A: Adder output
Channel B: sample-and-hold output

Place the output of the sample-and-hold channel on the input of a Tuneable LPF, which you will use as a reconstruction filter. Adjust the filter bandwidth to approximately 4 kHz (that is, about half of the sampling rate). Connect the output of the reconstruction filter to Channel A.
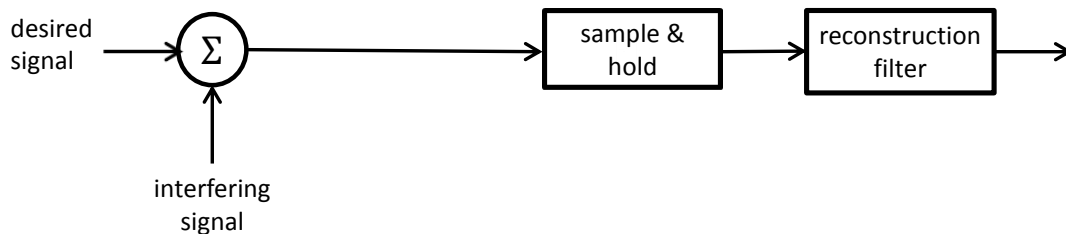
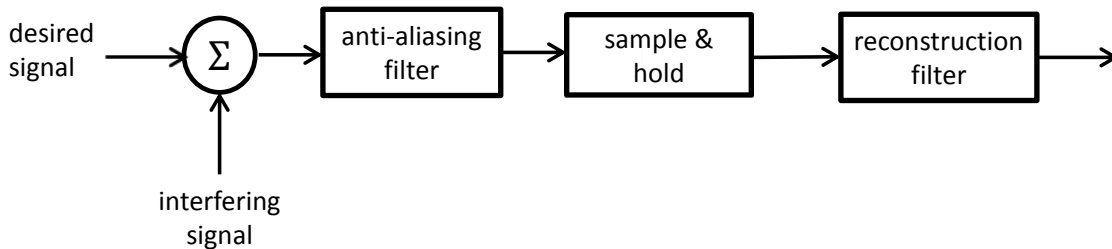**∿** Channel A: output of reconstruction filter

This should look like the sum of two sinusoids, but the higher-frequency (interfering) sinusoid will appear to have a lower frequency than before the composite signal was sampled.

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Select the Spectrum Mode. Observe the spectrum of the reconstruction filter output and note the two significant frequencies that are present. You should see that the 11.1-kHz interfering signal has been aliased to a lower frequency by the sampling process, and therefore this alias frequency is able to pass through the 4-kHz reconstruction filter.

📊   Channel A: output of reconstruction filter

The above exercise demonstrates that a relatively high-frequency interfering signal, with frequency greater than the bandwidth of the reconstruction filter, may nonetheless appear at the output of the reconstruction filter, having been aliased by the sampling operation to a lower frequency. Whenever sampling is done, there is a danger that some interfering signal, perhaps at a frequency quite far from that of the desired signal, will alias to the near vicinity of the desired signal. This problem is solved by introducing an anti-aliasing filter before the sampler.

Place the Adder output at the input of the other Tuneable LPF, which you will use as an anti-aliasing filter. Adjust the bandwidth of the anti-aliasing filter to approximately 4 kHz (that is, about half of the sampling rate). Place the output of this filter on the input of the sample-and-hold channel. The output of the sample-and-hold channel should still be connected to the reconstruction filter.

Observe the output of the reconstruction filter on first the oscilloscope and then the spectrum analyzer. You should find that the interfering signal has been removed by the anti-aliasing filter so that it could not be aliased to a lower frequency during the sampling operation.

〰️   Channel A: output of reconstruction filter (with anti-aliasing filter in place)

📊   Channel A: output of reconstruction filter (with anti-aliasing filter in place)