Math 116 Number Theory with Travis Kelm | **Homework #1 Introduction** | **Spring 2007 CSU - Fresno** | **Solutions**

1. **Shapes of Numbers -**

   (a) Find the first five numbers that are square and cubic. If you can, find a formula for the n^{th} such number. (Hint: How many prime factors in a square number? How many in a cubic number?)

   (b) 1 and 36 are two numbers that are triangular and square. Find three more or explain why three more do not exist.

Solution:

(a) Note that in a square number each prime factor must occur an even number of times. Similarly, in a cube, each prime factors must occur a multiple of three number of times. Since a number is a multiple of two and three if and only if the number is a multiple of six, we can conclude that a number n is a square and a cube exactly when each of the prime factors occurs a multiple of six number of times. This means that the number n is a sixth power. Thus the list of numbers that are squares and cubic is

$$\{1^6, 2^6, 3^6, 4^6, 5^6, \ldots, n^6, \ldots\}$$

(b) The formula for triangular numbers is $T_n = \dfrac{n(n+1)}{2}$ and the formula for square numbers is $S_k = k^2$. Thus we seek integers n and k such that

$$\frac{n(n+1)}{2} = k^2$$

Brute Force Approach: Using a computer you could simply generate a long list of the triangular numbers and then take the square root of each in order to see which triangulars are squares. However, since the fifth square-triangular number is the 1681th triangular number, this approach could take a long time. By using additional observations we can cut our work down quite a bit:

Note that T_n is a square if and only if either of the following are true:

Odd Type n is a square and $n+1$ is twice a square (Example: $n = 1$ or $n = 49$)

Even Type n is twice a square and $n+1$ is square (Example: $n = 8$ or $n = 288$)

Note that to be an **Odd Type** n must be an odd square and to be an **Even Type** n must be even.

Using a computer it is relatively easy to generate a list of many odd squares, and hence many candidates for **Odd Type** numbers. Each candidate can then be checked to see if $\dfrac{n+1}{2}$ is a square. Carrying out this program, the first few **Odd Type** numbers one finds are:

$$1 \qquad 49 \qquad 1681 \qquad 57,121$$

Similarly, using a computer it is relatively easy to generate a list of many even squares, double these squares, and then add one. Each of the resulting numbers can be checked to see if it is a square. Carrying out this program, the first few **Even Type** numbers one finds are:

$$8 \qquad 288 \qquad 9800 \qquad 332,928$$

Thus the first eight numbers that are triangular and square are:

$1\ (n=1)$	$36\ (n=8)$	$1225\ (n=49)$
$41,616\ (n=288)$	$1,413,721\ (n=1681)$	$48,024,900\ (n=9800)$
$1,631,432,881\ (n=57,121)$	$55,420,693,056\ (n=332,928)$	

Such numbers are quite rare indeed!

□

2. **Prime Triplets -** A set of three consecutive odd numbers that are all prime is called a prime triplet. For example, is a prime triplet. How many prime triplets are there?

   Solution: The only prime triplet is $\{3, 5, 7\}$. The reason for this is that in any set of three consecutive odd numbers, exactly one of the numbers is divisible by three. To see this, consider any set of three consecutive odd numbers. It will look like $\{2n + 1, 2n + 3, 2n + 5\}$ where $n \geq 1$. No matter what n is, it must have remainder 0, 1, or 2 when divided by 3. Let us simply consider each of those three cases:

   If $n/3$ has remainder 0 then $2n + 3$ is divisible by 3.

   If $n/3$ has remainder 1 then $2n + 1$ is divisible by 3.

   If $n/3$ has remainder 2 then $2n + 5$ is divisible by 3.

   Thus, because the only prime number that is divisible by three is the number 3, the only prime triplet is $\{3, 5, 7\}$. □

3. **Sum of Odds -** There is a formula for the sum of the first n odd numbers. Find it and prove it. (Hint: You can always use algebra, but there is a lovely geometric proof.)

   Solution: It is a lot of work to draw pictures with a computer so I will try to describe the geometric proof in words. Draw a 2×2 array of dots. Clearly there are $1 + 3 = 4$ dots. To get a 3×3 array from your 2×2 array you must

   (a) add one dot at the right-end of each row (since there are 2 rows that makes two added dots)
   (b) add one dot at the bottom of each column (since there are 2 rows that makes two added dots)
   (c) finally add one dot in the lower right corner

   Note that you added $2 \times 2 + 1 = 5$ dots. Now if you follow the same procedure to get a 4×4 array from your 3×3 array you will have to add $2 \times 3 + 1 = 7$ dots.

   From this you should see how you could give a geometric proof that the sum of the first n odd integers is in fact n^2. □

4. **Special Primes -** It is generally believed that there are infinitely many primes of the form $N^2 + 1$ where N is a natural number. (But no one has yet been able to prove it!) How many primes are there of the form $N^2 - 1$?

   Solution: Note that $N^2 - 1$ factors into $(N - 1)(N + 1)$ thus $N^2 - 1$ is a composite number anytime neither $(N - 1)$ nor $(N + 1)$ are equal to 1. The only time that $(N - 1)$ or $(N + 1)$ are equal to 1 is if $N = 2$ or $N = 0$. The $N = 0$ does not make sense in this situation, so we conclude that $N^2 - 1$ is prime only if $N = 2$ in which case $N^2 - 1 = 3$. Thus 3 is the only prime of the form $N^2 - 1$. □

5. **Odd Fibonaccis -** Find a simple formula for the sum of the first n Fibonacci numbers with odd indices. That is find a formula for

   $$F_1 + F_3 + F_5 + \cdots + F_{2n-1}$$

   Solution: The following equations provide a proof that

   $$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

$$\begin{aligned}
F_{2n} &= F_{2n-1} + F_{2n-2} & \\
&= F_{2n-1} + F_{2n-3} + F_{2n-4} & \text{since } F_{2n-2} = F_{2n-3} + F_{2n-4} \\
&= F_{2n-1} + F_{2n-3} + F_{2n-5} + F_{2n-6} & \text{since } F_{2n-4} = F_{2n-5} + F_{2n-6} \\
&\vdots & \\
&= F_{2n-1} + F_{2n-3} + F_{2n-5} + \cdots + F_5 + F_4 & \text{since } F_6 = F_5 + F_4 \\
&= F_{2n-1} + F_{2n-3} + F_{2n-5} + \cdots + F_5 + F_3 + F_2 & \text{since } F_4 = F_3 + F_2 \\
&= F_{2n-1} + F_{2n-3} + F_{2n-5} + \cdots + F_5 + F_3 + F_1 & \text{since } F_2 = F_1
\end{aligned}$$

□