Math 116 Number Theory with Travis Kelm	**Homework #2 Divisibility and Primes**	**Spring 2007 CSU - Fresno**	**Solutions**

In your solutions you must explain what you are doing using complete sentences.

Section 1.5

Exercise 8: Are there integers a, b, and $c \neq 0$ such that $a|bc$ but $a \nmid b$ and $a \nmid c$?

Solution: Oops, this question is repeated on Homework 3. I better not give you the answer until homework 3 is collected. □

Exercise 24: Find the number of positive integers not exceeding 1000 that are not divisible by 3 or 5.

Solution: From 1 to 1000, there are $999/3 = 333$ integers that are divisible by 3. There are $1000/5 = 200$ integers that are divisible by 5. Now how many are divisible by 3 **and** 5? Well, a number is divisible by 3 and 5 if and only if it is divisible by 15. There are $990/15 = 66$ number that are divisible by 15. Thus using the inclusion/exclusion principle, the number of integers from 1 to 1000 that are divisible by 3 **or** 5 is

$$333 + 200 - 66 = 467$$

This means that the number of integers from 1 to 1000 that are **not** divisible by 3 or 5 is $1000 - 467 = 533$. □

Section 3.1 - Prime Numbers

Exercise 2: Determine which of the following integers are prime:

201	207	213
203	211	221

Solution: For each number n we simply check whether n is divisible by any prime less than $\sqrt{n}$. Since the largest number is 221 and $\sqrt{221} \approx 14.8$ we need only check whether any of the numbers is divisible by 2, 3, 5, 7, 11, or 13. Following this program, one finds that:

$201 = 3 \times 67$	$207 = 3 \times 3 \times 23$	$213 = 3 \times 71$
$203 = 7 \times 29$	211 is prime	$221 = 13 \times 17$

□

Exercise 5: Find all the primes that are the difference of the fourth power of two integers.

Solution: We will prove that there are NO primes that are the difference of two fourth powers. That is, no prime can be written in the form $m^4 - n^4$ for $m, n \in \mathbb{N}$.

First, note that if $n \geq m$ then $m^4 - n^4$ is not positive so could not be prime. Thus we will assume that $m > n$.

If m, n are positive integers then

$$m^4 - n^4 = (m^2 + n^2)(m^2 - n^2) = (m^2 + n^2)(m + n)(m - n)$$

Thus the only way that $m^4 - n^4$ could be prime is if exactly two of the three factors $(m^2 + n^2), (m + n), (m - n)$ are equal to 1. Let us examine the possibilities.

If $m^2 + n^2 = 1$ then $m = 1$ and $n = 0$. Thus $m^4 - n^4 = 1$ so it is not prime. Thus the only possibility is that both $m + n = 1$ and $m - n = 1$. Thus $m + n = m - n$ so $2n = 0$ so $n = 0$. Thus $m^4 - n^4 = m^4$ which could not possibly be prime.

Thus, no matter how you slice it, $m^4 - n^4$ is not prime.

□

Proof of No Largest Prime - In the proof that there were an infinite number of primes we used the fact that if $p_1, p_2, \ldots, p_n$ is a list of primes then $p_1 p_2 \cdots p_n + 1$ is a number not divisible by any of the listed primes. However we never claimed that $p_1 p_2 \cdots p_n + 1$ is actually prime. Is it always prime? Observe that

- $2 + 1$
- $2 \times 3 + 1$
- $2 \times 3 \times 5 + 1$
- $2 \times 3 \times 5 \times 7 + 1$

are all prime. However, if you follow this pattern you will find a composite number eventually. Find the three smallest composite numbers of the form $p_1 p_2 \cdots p_n + 1$.

Solution: By continuing to try examples one quickly finds that

$$\begin{aligned} 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 &= 59 \times 509 \\ 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 + 1 &= 19 \times 97 \times 277 \\ 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 + 1 &= 347 \times 27953 \end{aligned}$$

Note that you don't see another prime until

$$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 + 1$$

An interesting question is how many primes are there that can be written in this form? (If you could answer this question you would become rather famous, since no one currently knows the answer.) $\square$