

In your solutions you must explain what you are doing using complete sentences.

Section 3.3 - Greatest Common Divisors

Exercise 2: Find the GCD of the following pairs:

$5, 15$

$-27, -45$

$100, 121$

$0, 100$

$-90, 100$

$1001, 289$

Solution:

1. Since $5|15$ we have that $\gcd(5, 15) = 5$.
2. Since $100|0$ we have that $\gcd(0, 100) = 100$.
3. $\gcd(-27, -45) = 9$ since $-27 \nmid -45$ and 9 is the largest factor of 27 besides 27 itself.
4. The prime factorization of 90 is $2 \cdot 3^2 \cdot 5$ and the prime factorization of 100 is $2^2 \cdot 5^2$. Thus $\gcd(-90, 100) = 10$.
5. Since $121 = 11^2$ and $11 \nmid 100$ we have that $\gcd(121, 100) = 1$.
6. Using the Euclidean algorithm, one finds that $\gcd(1001, 289)$ is 1.

□

Exercise 6: Let a be a positive integer. What is the GCD of a and $a + 2$?

Solution: Since the gcd is the smallest possible positive linear combination of a and $a + 2$, we must have that $\gcd(a, a + 2) \leq 2$ since $2 = 1(a + 2) - 1(a)$. If a is even, then so is $a + 2$, thus 2 does divide both a and $a + 2$ so we must have that $\gcd(a, a + 2) = 2$. But if a is odd then 2 cannot be a divisor of a thus we must have that $\gcd(a, a + 2) = 1$. □

Exercise 18: Find three mutually relatively prime integers from among the integers 66, 105, 42, 70, and 165.

Solution: The prime factorizations are

- $66 = (2)(3)(11)$
- $105 = (3)(5)(7)$
- $42 = (2)(3)(7)$
- $70 = (2)(5)(7)$
- $165 = (3)(5)(11)$

Thus a set of mutually relatively prime numbers is $\{66, 105, 70\}$. Another is $\{42, 70, 165\}$.

□

Section 3.4 - The Euclidean Algorithm

Exercise 2: Use the Euclidean Algorithm to find the GCD of each of the following pairs:

$51, 87$

$981, 1234$

$105, 300$

$34709, 100313$

Solution: See the solution to Exercise 4 below.

□

Exercise 4: For each of the pairs in Exercise 2 write the GCD as a linear combination of the two integers.

Solution: One can use the Euclidean Algorithm to find each of the following:

- $\gcd(87, 51) = 3 = (-7)87 + (12)51$
- $\gcd(300, 105) = 15 = (-1)300 + (3)105$
- $\gcd(1234, 981) = 1 = (190)1234 + (-239)981$
- $\gcd(100313, 34709) = 1 = (2175)100313 + (-6286)34709$

□

Section 3.5 - The Fundamental Theorem of Arithmetic

Exercise 30: Find the LCM of each of the following pairs:

8, 12	28, 35	256, 5040
14, 15	111, 303	343, 999

Solution:

1. We have $\gcd(8, 12) = 4$ so by the GL theorem $\text{lcm}(8, 12) = \frac{8 \times 12}{4} = 24$.
2. We have $\gcd(14, 15) = 1$ so by the GL theorem $\text{lcm}(14, 15) = \frac{14 \times 15}{1} = 210$.
3. We have $\gcd(28, 35) = 7$ so by the GL theorem $\text{lcm}(28, 35) = \frac{28 \times 35}{7} = 140$.
4. We have $\gcd(111, 303) = 3$ so by the GL theorem $\text{lcm}(111, 303) = \frac{111 \times 303}{3} = 11,211$.
5. We have $\gcd(5040, 256) = 16$ so by the GL theorem $\text{lcm}(5040, 256) = \frac{5040 \times 256}{16} = 80,640$.
6. We have $\gcd(999, 343) = 1$ so by the GL theorem $\text{lcm}(999, 343) = \frac{999 \times 343}{1} = 342,657$.

□

Exercise 35: One kind of cicada emerges every 17 years. Another kind emerges every 13 years. If they both emerge in the year 1900, what is the next year that they will both emerge in again?

Solution: We seek the LCM of 13 and 17. Since they are both prime, the LCM is the product $17 \times 13 = 221$. Thus they won't both emerge in the same year for 221 years, or the year 2121. □

Exercise 36: Which pairs of integers a and b have $\gcd 18$ and $\text{lcm} 540$?

Solution: Since 18 is a common divisor we have that there are integers α and β such that:

$$a = 2 \times 3^2 \times \alpha \qquad b = 2 \times 3^2 \times \beta$$

Thus clearly, $ab = 2^2 \times 3^4 \times \alpha \times \beta$

By the GL theorem $ab = 18 \times 540 = 2^3 \times 3^5 \times 5$. This gives $2^2 \times 3^4 \times \alpha \times \beta = 2^3 \times 3^5 \times 5$. Thus we must have that

$$\alpha\beta = 2 \times 3 \times 5$$

So the question becomes, how many ways can we divide up three primes among two integers? The following table shows the four possibilities for α and β and the corresponding a and b :

α	β	a	b
$2 \times 3 \times 5$	1	540	18
3×5	2	270	36
2×5	3	180	54
2×3	5	108	90

□

Exercise 40: Use Lemma 3.4 to show that if p is prime and $a \in \mathbb{Z}$ such that $p|a^2$, then $p|a$.

Solution: We will prove the contrapositive. Assume that $p \nmid a$. We must show that $p \nmid a^2$. Since p is prime and $p \nmid a$ we must have that $\gcd(a, p) = 1$. Thus by Lemma 3.4, if $p|a \times a$ then we would have $p|a$ which is definitely not the case. Thus we must have that $p \nmid a^2$. \square

Exercise 50: Find the two positive integers with sum 798 and LCM 10,780.

Solution: By the fact that $\gcd(x, y) = \gcd(x + y, \text{lcm}(x, y))$ we have that $\gcd(x, y) = \gcd(798, 10780) = 14$. So by the GL theorem we have that $xy = 14 \times 10780 = 150,920$. Since we know that $x + y = 798$ we can make the substitution $y = 798 - x$. This gives

$$x(798 - x) = 150,920$$

This is a quadratic equation with solutions $x_1 = 308$ and $x_2 = 490$. Thus it follows that the positive integers that we are looking for are $x = 308$ and $y = 490$. \square

Exercise 64: A company sells \$375,961 of a book. If the price of the book is an exact dollar amount greater than \$1, how many copies did they sell?

Solution: We have that $\$375,961 = (\text{price of book}) \times (\# \text{ books sold})$. The prime factorization of 375,961 is 79×4759 . Thus either the book costs \$79 and 4759 copies were sold, or the book costs \$4759 and 79 copies were sold. \square

Exercise 68: Show that if a_1, a_2, \dots, a_n are pairwise relatively prime, then $\text{lcm}(a_1, a_2, \dots, a_n) = a_1 a_2 \dots a_n$

Solution: First notice that if a, b are relatively prime, then by the GL theorem $\text{lcm}(a, b) = \frac{ab}{\gcd(a, b)} = ab$. Thus the result is certainly true for $n = 2$.

Now suppose that the result is true for sets of size $n - 1$. That is, assume that

$$\text{lcm}(a_1, a_2, \dots, a_{n-1}) = a_1 a_2 \dots a_{n-1}$$

Now if a_n is prime to a_1, a_2, \dots, a_{n-1} then there can be no prime factors of a_n in the product $a_1 a_2 \dots a_{n-1}$, thus the smallest multiple of both a_n and $a_1 a_2 \dots a_{n-1}$ is $a_1 a_2 \dots a_{n-1} a_n$. But by assumption $a_1 a_2 \dots a_{n-1}$ is the smallest multiple of a_1, a_2, \dots, a_{n-1} , thus $\text{lcm}(a_1, a_2, \dots, a_n) = a_1 a_2 \dots a_n$ as desired. \square

Questions Not in Book

Divisors of Products - Is it true that if $a|bc$ then $a|b$ or $a|c$?

1. Show that the answer to the above question is NO by finding three positive integers $a < b$ and $a < c$ with $a|bc$ but with $a \nmid b$ and $a \nmid c$.

Solution: There are many answers to this. One of the simplest is $a = 4$ $b = 6$ $c = 10$. \square

2. Prove that if $a|bc$ and if $\gcd(a, b) = 1$ then $a|c$.

Solution: If $a = 1$ then $a|c$ is obvious. So we might as well assume that $a \neq 1$. By the Fund Theorem of Arith. a has a unique prime decomposition. Since $\gcd(a, b) = 1$, none of the primes in a can be in the prime decomp of b . Thus *all* of the primes in the prime decomposition of a must also be in the prime decomposition of c . It follows that a divides c . \square

Ten Divisors - Characterize the integers with exactly 10 divisors and give a few examples of each type.

Solution: There are two ways two write ten as the product of positive integers, namely 2×5 and 10 by itself, thus if $n \in \mathbb{Z}$ has exactly ten divisors, the prime decomposition of n must be either

$$n = pq^4 \quad \text{or} \quad n = p^9$$

where p and q are distinct primes. A few examples of the p^9 type are

$$512 \quad 19,683 \quad 1,953,125$$

A few examples of the pq^4 type are

$$48 \quad 80 \quad 162 \quad 1250$$

\square