

Section 7.2 - The Sum and Number of Divisors

1. From Class: Show that if m and n are relatively prime, then $\tau(m \cdot n) = \tau(m) \cdot \tau(n)$.

Solution: Let

$$m = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$$

$$n = q_1^{r_1} q_2^{r_2} \cdots q_l^{r_l}$$

be the prime factorizations of m and n . If m and n are relatively prime, then none of the p 's are equal to any of the q 's. This implies that the prime factorization of mn is simply

$$mn = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k} q_1^{r_1} q_2^{r_2} \cdots q_l^{r_l}$$

and hence

$$\tau(mn) = (t_1 + 1)(t_2 + 1) \cdots (t_k + 1)(r_1 + 1)(r_2 + 1) \cdots (r_l + 1)$$

But note that based upon the prime factorizations of m and n we have:

$$\tau(m) = (t_1 + 1)(t_2 + 1) \cdots (t_k + 1)$$

$$\tau(n) = (r_1 + 1)(r_2 + 1) \cdots (r_l + 1)$$

Thus we've shown that $\tau(m \cdot n) = \tau(m) \cdot \tau(n)$. \square

2. Exercise 7: Fix a positive integer k . Show that the equation $\tau(n) = k$ has infinitely many solutions.

Solution: If p is any prime, then $n = p^{k-1}$ is a solution to $\tau(n) = k$. Since there are infinitely many primes, this gives infinitely many solutions. \square

3. Exercise 11: $\sigma(n)$ is the *sum* of all positive divisors of n . Find a prove a formula for the *product* of all positive divisors of n .

Solution: Let $D = \{1, d_2, d_3, \dots, n\}$ be the set of all divisors of n ordered from smallest (i.e. 1) to largest (i.e. n). Note that the set D has $\tau(n)$ elements. We wish to determine the product $1d_2d_3 \cdots n$. A little sneakiness will pay off here. Observe that for any integer, n , the smallest divisor, 1, times the largest divisor, n , is of course equal to n . But in addition, the *second* smallest divisor, times the *second* largest divisor is also equal to n . And guess what? The *third* smallest divisor times the *third* largest divisor is also n . And so on. So consider the following sneaky algebra. In the array below multiply down along each column and then multiply along the third row:

$$\begin{array}{cccccc} 1 & d_1 & d_2 & \cdots & d_{\tau(n)-1} & n \\ n & d_{\tau(n)-1} & d_{\tau(n)-2} & \cdots & d_1 & 1 \\ \hline n & n & n & \cdots & n & n \end{array} \longrightarrow n^{\tau(n)}$$

But if you look hard enough you'll see that the product computed in the array is also $(1d_2d_3 \cdots n)^2$. Thus

$$(1d_2d_3 \cdots n)^2 = n^{\tau(n)}$$

and hence

$$(1d_2d_3 \cdots n) = \sqrt{n^{\tau(n)}} = n^{\tau(n)/2} \quad \square$$

4. Exercise 12: Fix a positive integer k . Show that if the equation $\sigma(n) = k$ has any solutions, then it has only a finite number of solutions.

Solution: For all integers k we have that $1, k$ are divisors of k , thus $\sigma(k) \geq k + 1$. Hence the only possible solutions to $\sigma(n) = k$ must be less than k . Since there are only $k - 1$ positive integers less than k , it follows that the equation $\sigma(n) = k$ has no more than $k - 1$ solutions. \square

Section 7.3 - Perfect Numbers and Mersenne Primes

5. Exercise 2: Find the seventh and eighth even perfect numbers.

Solution: From the table on page 264 of the book, the seventh and eighth Mersenne primes are:

- $2^{19} - 1 = 524,287$
- $2^{31} - 1 = 2,147,483,647$

Thus the seventh and eighth perfect numbers are:

- $2^{18}(2^{19} - 1) = 137,438,691,328$
- $2^{30}(2^{31} - 1) = 2,305,843,008,139,952,128$

Alternate Solution: The first eight perfect numbers are

1. 6
2. 28
3. 496
4. 8,128
5. 33,550,336
6. 8,589,869,056
7. 137,438,691,328
8. 2,305,843,008,139,952,128

The following is the MAPLE code that one could use to find them

```
myList:=[]:
for i from 1 to 50 do
  if isprime(2^i-1) then
    myList:=[op(myList),2^(i-1)*(2^i-1)]
  end if:
end do:
print(myList);
```

As you might observe the program utilizes Euler's characterization of even perfect numbers (Theorem 7.10 in our book.). \square

6. Exercise 4: Find a factor of each of the following integers:

(a) $2^{111} - 1$

Solution: Since $111 = 3 \times 37$ we have that

$$2^{111} - 1 = (2^{37})^3 - 1 = (2^{37} - 1)((2^{37})^2 + 2^{37} + 1)$$

(b) $2^{289} - 1$

Solution: Since $289 = 17 \times 17$ we have that

$$2^{289} - 1 = (2^{17})^{17} - 1 = (2^{17} - 1)((2^{37})^{16} + (2^{17})^{15} + \dots + 1)$$

(c) $2^{46189} - 1$

Solution: Since $46189 = 11 \times 4199$ we have that

$$2^{46189} - 1 = (2^{11})^{4199} - 1 = (2^{11} - 1)((2^{11})^{4198} + (2^{11})^{4197} + \dots + 1) \square$$

Abundant and Deficient Numbers A positive integer n is called *abundant* if $\sigma(n) > 2n$ and is called *deficient* if $\sigma(n) < 2n$. See if you can answer the following questions about such numbers. *Hint:* It is ok to use the results of other exercises (even ones that were not assigned) to help solve these problems.

7. Exercise 5: Find the six smallest abundant integers.

Solution: They are 12, 18, 20, 24, 30, and 36. One can use a computer to find them. Below is the MAPLE code.

```
myList:=[]:
for z from 2 by 2 to 50 do
  if sigma(z)>2*z then
    myList:=[op(myList),z]
  end if:
end do:
print(myList);
```

8. Exercise 6: Find the smallest odd abundant integer.

Solution: The first odd abundant number does not occur until is 945. You can use a computer to find it. Below is the MAPLE code.

```
myList:=[]:
for z from 3 by 2 to 1000 do
  if sigma(z)>2*z then
    myList:=[op(myList),z]
  end if:
end do:
print(myList);
```

9. Exercise 10: Show that if $n = 2^{m-1}(2^m - 1)$ where $m > 0$ and $2^m - 1$ is composite, then n is abundant.

Solution: Note first that 2^{m-1} and $2^m - 1$ are relatively prime since the first is a power of 2 and the other is odd. Thus we know that $\sigma(n) = \sigma(2^{m-1})\sigma(2^m - 1)$. Using the rule for powers of primes, we know that $\sigma(2^{m-1}) = 2^m - 1$. Next, since $2^m - 1$ is composite we must have that $\sigma(2^m - 1) > 2^m$. Thus we have

$$\sigma(n) = \sigma(2^{m-1})\sigma(2^m - 1) > (2^m - 1)2^m = 2(2^{m-1}(2^m - 1)) = 2n$$

This proves that n is abundant. \square

10. Exercise 12-13: Show that there are infinitely many even abundant numbers and infinitely many odd abundant numbers.

Solution: By exercise 9 in the book any multiple of an abundant number is also abundant. 12 is an even abundant number, so any multiple of 12 is an abundant even number. 945 is an abundant odd number, so any odd multiple of 945 is an abundant odd number. \square